Gaussian and non-Gaussian finite-size scalings of the spread-out Ising model at the free-boundary pseudocritical point

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Abstract

For the critical weakly coupled $|\varphi|^4$ model with periodic boundary conditions, Park [10] showed that only the constant mode has enhanced fluctuations. Motivated by the shifted free-boundary critical window established by Michta, Park, and Slade [13] for the hierarchical model, we study the spread-out Ising model at its free-boundary pseudocritical point. At the upper critical dimension we determine the exponential moments of the spin field tested against smooth real-valued functions vanishing on the boundary. For a nonzero test, the spin sum divided by its standard deviation has a quartic limiting law if the test overlaps the ground state and a standard Gaussian limit if it is orthogonal. In four dimensions the orthogonal susceptibility has order n^2 in a box of side n, the scale obtained by Camia, Jiang, and Newman [11] at bulk criticality under a mean-field two-point hypothesis. The lowest sine and the $(3, 1, 1, 1)$ sine have different limiting laws, so it provides a counterevidence to the all-odd classification of Flores-Sola, Berche, Kenna, and Weigel [2].

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1 The model and the two susceptibility scales

We consider the spread-out Ising model on a box with free boundary condition: fix $d \in \{1, 2, 3, 4\}$, $\alpha = d/4$, $s = 2\alpha = d/2$, $\Lambda_n = \{1, \dots, n\}^d$ and $V = n^d$. Let D_R be the probability measure uniform

on $0 < |z|_\infty \le R$ in $\mathbb{Z}^d$, and set $L_R = I - D_R$. For inverse temperature $\tau > 0$, we consider the measure on $\sigma \in \{-1,1\}^{\Lambda_n}$ with weight proportional to

$$\exp\{-\tfrac{1}{2}\tau\, \sigma^{\mathsf{T}} J_n \sigma\}, \qquad J_n = P_{\Lambda_n} L_R^{\alpha} P_{\Lambda_n}^{*}. \tag{1.1}$$

where $P_{\Lambda_n} : \ell^2(\mathbb{Z}^d) \to \ell^2(\Lambda_n)$ is the simple restriction to Λ_n and $P_{\Lambda_n}^{*}$ extends by zero, so the fractional power is taken before restriction to the box. Fix R sufficiently large. For $n \ge n_0(R)$ choose the pseudocritical point $\tau = \tau_{\Lambda_n}^{\mathrm{pc}}$ constructed in [1]. At this temperature the negative logarithm of the marginal density of the prescribed terminal coordinate has zero curvature at the origin. Section 2 of [1] defines its principal direction. Flores-Sola et al. describe their pseudocritical temperature T_L using a finite-volume response peak [2].

At the free-boundary pseudocritical point, the variance of the total spin divided by the volume is of order $n^{d/2}\sqrt{\log n}$ in a box of side n [1]. Does the second moment carry the factor $\sqrt{\log n}$ for any test function? The question comes from Table I of Flores-Sola, Berche, Kenna and Weigel [2]. At $d = d_c$ we find that the logarithmic enhancement is determined by overlap with the ground state. In the short-range model this singles out the lowest sine. In the long-range models the ground state is a superposition of infinitely many sines. The difference between overlap with the uniform field and overlap with the ground state can be seen explicitly in (1.22).

Unless an inverse temperature is displayed explicitly, spin expectations are taken at the pseudocritical point.

Definition 1.1 (Ising measure at $\tau_{\Lambda_n}^{\mathrm{pc}}$)**.** *Write $\mathbb{E}_{\mathrm{sp}}$ for expectation under this spin law given by the weight* (1.1) *and $G_{\mathrm{sp}}(x,y) = \mathbb{E}_{\mathrm{sp}}[\sigma_x \sigma_y]$ for its two-point function having* **fixed** $\tau = \tau_{\Lambda_n}^{\mathrm{pc}}$.

We start with a remark that $\tau_{\Lambda_n}^{\mathrm{pc}}$ is a genuine shift from the bulk cricial point $\tau_c = \tau_c(d,R) = \inf\{\tau > 0 : \langle\sigma_0\rangle_\tau^+ > 0\}$, where $\langle\cdot\rangle_\tau^+$ is the infinite-volume plus state for the interaction in (1.1). Write $\Delta_n = \tau_{\Lambda_n}^{\mathrm{pc}} - \tau_c$ and $(u)_+ = \max\{u, 0\}$.

Theorem 1.2 (Shift of the pseudocritical point)**.** *For $d \in \{1,2,3,4\}$ and sufficiently large fixed R, $\tau_{\Lambda_n}^{\mathrm{pc}} \to \tau_c$ as $n \to \infty$. For $n \ge n_1(d,R)$ one has $(\Delta_n)_+ \le C_d R^{-d}$. For $d = 1,2,3$ one also has*

$$(\Delta_n)_+ \le C_{d,R} \begin{cases} n^{-1/10}(\log n)^{1/10}, & d = 1,3, \\ n^{-1/5}(\log n)^{1/2}, & d = 2. \end{cases} \tag{1.2}$$

There is $c_{d,R} > 0$ such that, for all sufficiently large n,

$$\Delta_n \ge c_{d,R}\, n^{-s} = c_{d,R}\, n^{-d/2}. \tag{1.3}$$

Refinement of this theorem is of seperate interest. Since the theorem is independent of the rest of the article, we isolate the proof in Appendix C.

Spin observables. All lattice and continuum test functions are real-valued. For lattice vectors we use the counting-measure scalar product $\langle f, h\rangle = f^{\mathsf{T}} h$. The spin sum is $S_n(f) = \langle f, \sigma\rangle = \sum_{x\in\Lambda_n} f(x)\sigma_x$. On $\Omega = (0,1)^d$, define the continuum sines for $k \in \mathbb{N}^d$, where $\mathbb{N} = \{1,2,\ldots\}$, and their lattice samples for $1 \le k_i \le n$ by

$$e_k(z) = 2^{d/2} \prod_{i=1}^{d} \sin(\pi k_i z_i), \qquad f_n(F)(x) = (n+1)^{-d/2} F(x/(n+1)), \qquad \phi_{k,n} = f_n(e_k). \tag{1.4}$$

The lattice sines are orthonormal and vanish on the Dirichlet planes $x_i = 0, n+1$. Write $k_g = (1,\dots,1)$, $e_g = e_{k_g}$ and $\phi_{g,n} = \phi_{k_g,n}$. The covariance and F-susceptibility of sampled tests are

$$\mathcal{C}_n(F,H) = \mathbb{E}_{\mathrm{sp}}[S_n(f_n(F))S_n(f_n(H))] = f_n(F)^{\mathsf{T}} G_{\mathrm{sp}} f_n(H), \qquad \chi_n(F) = \mathcal{C}_n(F,F). \tag{1.5}$$

Thus $\chi_{k,n} = \chi_n(e_k)$ is the susceptibility of a unit sine, and the lattice test $V^{-1/2}\mathbf{1}$ gives the uniform susceptibility.

The continuum operator. Let $\Omega = (0,1)^d$ be equipped with Lebesgue measure. Let H_α be the self-adjoint operator of the closed form

$$\mathcal{E}_\alpha(F,F) = \int_{\mathbb{R}^d} |\xi|^{2\alpha} |\widehat{\widetilde{F}}(\xi)|^2 \, d\xi, \qquad \widetilde{F} \in H^\alpha(\mathbb{R}^d), \quad \widetilde{F} = 0 \text{ off } \overline{\Omega}, \tag{1.6}$$

where $\widetilde{F}$ is the zero extension of F and $\widehat{u}(\xi) = (2\pi)^{-d/2} \int e^{-ix\cdot\xi} u(x)\, dx$. Write $\mu_1 < \mu_2 \le \cdots$ for the eigenvalues, counted with multiplicity, and w for the nonnegative L^2-unit ground state. The reduced resolvent $\mathcal{R}_\alpha$ inverts $H_\alpha - \mu_1$ on $w^\perp$. We use

$$\mathcal{R}_\alpha w = 0, \qquad \mathcal{R}_\alpha|_{w^\perp} = (H_\alpha - \mu_1)^{-1}|_{w^\perp}, \qquad I_4 = \int_\Omega w^4, \qquad b_F = \langle w, F\rangle. \tag{1.7}$$

For sine tests write $b_k = b_{e_k}$ and $b_g = b_{e_g} > 0$. The positivity follows from $e_g > 0$ in Ω and $w \ge 0$, $\|w\|_2 = 1$. In dimension four H_1 is the ordinary Dirichlet Laplacian. For $0 < \alpha < 1$, (1.6) defines the restricted fractional Laplacian. We call the principal direction the non-Gaussian mode and the directions in $w^\perp$ the Gaussian modes. Write $g = g_n \asymp_{d,R} (\log V)^{-1}$ for the terminal quartic coefficient at the matched point. Let W_4 be a real random variable with density proportional to $e^{-x^4/4}$. Its Laplace transform and variance are

$$\Phi_4(t) = \frac{\int_{\mathbb{R}} e^{tx - x^4/4}\, dx}{\int_{\mathbb{R}} e^{-x^4/4}\, dx}, \qquad v_4 = \mathbb{E}W_4^2 = \frac{2\Gamma(3/4)}{\Gamma(1/4)}. \tag{1.8}$$

Theorem 1.3 (Susceptibility and exponential moments)**.** *Let $F \in C^\infty(\overline{\Omega};\mathbb{R})$ vanish on the faces and set $X = S_n(f_n(F))$. With the positive spin amplitudes in* (3.3)*, the following asymptotics hold as $n \to \infty$ at fixed sufficiently large R. The exponential-moment estimates are uniform for θ in bounded real intervals.*

- Non-Gaussian scaling. *If $b_F \neq 0$, then*

$$\chi_n(F) = \frac{n^s}{\sqrt{g}} A_n^{\mathrm{NG}} b_F^2 (1 + o(1)). \tag{1.9}$$

$$\mathbb{E}_{\mathrm{sp}} \exp\Bigl(\theta g^{1/4} n^{-s/2} X\Bigr) = \Phi_4\Bigl(\theta b_F \sqrt{A_n^{\mathrm{NG}}/v_4}\Bigr) + o(1). \tag{1.10}$$

- Gaussian scaling. *If $b_F = 0$, then*

$$\chi_n(F) = n^s A_n^{\mathrm{G}} \langle F, \mathcal{R}_\alpha F\rangle + o(n^s). \tag{1.11}$$

$$\mathbb{E}_{\mathrm{sp}} \exp\Bigl(\theta n^{-s/2} X\Bigr) = \exp\Bigl(\tfrac{1}{2}\theta^2 A_n^{\mathrm{G}} \langle F, \mathcal{R}_\alpha F\rangle\Bigr) + o(1). \tag{1.12}$$

This theorem is an analogue of [10, Theorem 1.9]. The moments here are computed at $\tau^{\mathrm{pc}}_{\Lambda_n}$. Appendix C shows that this inverse temperature converges to the bulk critical point and lies at least $c_{d,R}n^{-d/2}$ above it for all sufficiently large n.

For $F \neq 0$, division by the spin standard deviation removes the possibly n-dependent amplitude. Theorem 1.3 then gives

$$\frac{X}{\sqrt{\chi_n(F)}} \xrightarrow{\text{law}} \begin{cases} W_4/\sqrt{v_4}, & b_F \neq 0, \\ N(0,1), & b_F = 0. \end{cases} \tag{1.13}$$

The symmetry of W_4 removes the sign of b_F in the first limit.

Two tests can couple to the same non-Gaussian mode. To determine their covariance to accuracy $o(n^s)$, we retain the finite-volume susceptibility $\chi_{k_g,n}$ in the principal term. Its relative asymptotic in (1.9) alone allows an error larger than n^s. Put

$$j_F = F - \frac{b_F}{I_4} w^3, \qquad \mathcal{B}(F,H) = \langle j_F, \mathcal{R}_\alpha j_H \rangle, \qquad r_F = b_F/b_g. \tag{1.14}$$

The map $F \mapsto j_F$ is the projection onto $w^\perp$ along w^3, since $\langle w, w^3 \rangle = I_4$. The direction w^3 comes from the first derivative of the quartic potential.

Theorem 1.4 (Mixed moments at the Gaussian scale)**.** *For fixed smooth F, H vanishing on the faces as in Theorem 1.3,*

$$\mathcal{C}_n(F,H) = r_F r_H \, \chi_{k_g,n} + n^s A_n^{\mathrm{G}} \left[\mathcal{B}(F,H) - r_F r_H \mathcal{B}(e_g, e_g) \right] + o(n^s). \tag{1.15}$$

In particular, if $b_F = b_H = 0$, then $n^{-s}\mathcal{C}_n(F,H) = A_n^{\mathrm{G}} \langle F, \mathcal{R}_\alpha H \rangle + o(1)$.

1.1 Related works

Temperature shift and rounding. Fisher and Barber distinguished the displacement of a finite-system transition from the width of the temperature interval over which it is rounded [3]. The distinction matters for a free box when the bulk critical temperature lies outside the finite-volume critical window. For periodic systems, Brézin and Zinn-Justin integrated out the nonconstant modes to obtain an effective Hamiltonian for the spatial average [4]. Its quartic term controls the fluctuations of the average even when the renormalised coupling tends to zero. Rudnick, Gaspari and Privman used standing waves to analyse the shift of the transition and the susceptibility in a free Ising box above four dimensions [5]. In simulations of the five-dimensional Ising model, Wittmann and Young found free-box susceptibility scaling as L^2 at the bulk critical temperature and as $L^{5/2}$ near the finite-volume pseudocritical temperature, where L denotes the side length [6].

Role of the Fourier modes on the finite-size scaling. Wittmann and Young proposed that the all-odd sines inherit the scaling of the uniform magnetization through their nonzero spatial means. Their mode susceptibility measurements included the index $(2,1,1,1,1)$, whose spatial mean vanishes [6, Sec. V.B]. Flores-Sola et al. gave the prediction separately for each boundary condition and temperature [2, Table I]. Write s for their dispersion exponent σ, with $s = 2$ for short-range interactions. For $d > 2s$ and free boundaries at T_L, they assign all-odd indices to the Q sector, with susceptibility of order $L^{d/2}$. An even entry places the index in the Gaussian (G) sector, with susceptibility of order L^s. At the bulk critical temperature their free-boundary row gives Gaussian scaling for both classes. Their figures compare the total magnetization and the lowest G modes in five-dimensional short-range and one-dimensional long-range Ising models. The total magnetization alone cannot distinguish the contributions of two individual all-odd sines.

Honchar, Berche, Holovatch and Kenna subsequently compared the two all-odd indices $(1,1,1,1,1)$ and $(3,1,1,1,1)$ on a five-dimensional free box [7, Fig. 7]. They took sine coefficients of a magnetization profile formed by aligning each spin configuration with the sign of its total magnetization. The fitted slopes were -1.26 and -1.41, respectively, and the authors left the asymptotic interpretation of the mode data open. The quantity $\chi_{k,n}$ studied here is the second moment of the sine coefficient of a spin configuration before alignment.

Logarithms at the upper critical dimension. At $d = 2s$ the two powers printed in Table I coincide. The quartic fluctuation can nevertheless be detected by its multiplicative logarithm. For periodic long-range Ising models, Luijten and Blöte found the predicted logarithmic correction to the uniform susceptibility at the marginal decay exponent [8]. Lv, Xu, Sun, Chen and Deng measured the distinction directly in periodic four-dimensional $O(N)$ models with $N = 1, 2, 3$. The uniform susceptibility grew as $L^2\sqrt{\log L}$ and the sampled nonzero Fourier modes grew as L^2 [9]. For the weakly coupled $|\varphi|^4$ model on a Euclidean torus, Park proved the distinction between the zero and nonzero Fourier susceptibilities, including the marginal logarithm [10, Corollary 1.10]. Translation invariance makes the constant function the principal mode in these periodic examples. The free-box calculation requires its spatial profile.

Rigorous results for free boxes and critical windows. For the nearest-neighbour Ising model in $d > 4$, Camia, Jiang and Newman deduced order L^2 for the free-box susceptibility at the bulk critical temperature from mean-field decay of the infinite-volume two-point function [11]. The bounds of Duminil-Copin and Panis establish that decay for every $d \geq 5$ [12]. For the hierarchical $|\varphi|^4$ model, Michta, Park and Slade proved that the spatial average has the same quartic scaling law in the periodic and free-boundary windows about their respective effective critical points. Above four dimensions the free-boundary shift has order L^{-2} and the window width has order $L^{-d/2}$ [13]. Park and Slade also obtained the two-point plateau in these windows, including the logarithmic correction in four dimensions [14]. For Euclidean lattices, Liu, Park and Slade connect the periodic susceptibility and the plateau to estimates on the infinite lattice. Their Conjecture 6 concerns the susceptibility and two-point profiles in a shifted free-boundary window above the upper critical dimension [15]. At the matched temperature studied here, Theorems 1.3 and 1.4 give the spatial profile of the enhanced fluctuation and the covariance on its orthogonal complement.

1.2 Sine modes and the orthogonality condition

For every n and every lattice test f, Fourier expansion in the Dirichlet sine basis gives the exact identities

$$M_{kl}^{(n)} = \sum_{x,y\in\Lambda_n} \phi_{k,n}(x) G_{\mathrm{sp}}(x,y) \phi_{l,n}(y), \qquad \chi_{k,n} = M_{kk}^{(n)}, \tag{1.16}$$

$$f = \sum_k d_k \phi_{k,n}, \qquad d_k = \langle \phi_{k,n}, f \rangle, \qquad \mathbb{E}_{\mathrm{sp}}[S_n(f)^2] = \sum_{k,l} d_k M_{kl}^{(n)} d_l. \tag{1.17}$$

If $f \perp \phi_{g,n}$ the sum in (1.17) ranges over $k, l \neq k_g$. Reflection of coordinate i about $(n+1)/2$ multiplies the kth sine by $(-1)^{k_i+1}$ and preserves the spin law. Consequently $M_{kl}^{(n)} = 0$ whenever k_i, l_i have different parity for some i. The ground state w is invariant under these reflections by simplicity. Thus $b_k = 0$ if any k_i is even. Write $F_k = \langle e_k, F \rangle$ for the continuum sine coefficients and $|k|^2 = \sum_{i=1}^d k_i^2$.

Corollary 1.5 (Tests orthogonal to the lowest sine)**.** *In $d = 4$, we have $w = e_g$, $I_4 = (3/2)^d$, and $b_g = 1$. For each fixed $k \neq k_g$ the susceptibility is*

$$\chi_{k,n} = \frac{n^2 A_n^{\mathrm{G}}}{\pi^2(|k|^2 - d)} + o(n^2). \tag{1.18}$$

For all fixed $k, l \neq k_g$, including distinct indices in a degenerate eigenspace,

$$M_{kl}^{(n)} = n^2 A_n^{\mathrm{G}} \frac{\delta_{kl}}{\pi^2(|k|^2 - d)} + o(n^2). \tag{1.19}$$

If $F \perp e_g$ is as in Theorem 1.3, its susceptibility is

$$\chi_n(F) = n^2 A_n^{\mathrm{G}} \sum_{k \neq k_g} \frac{F_k^2}{\pi^2(|k|^2 - d)} + o(n^2). \tag{1.20}$$

In dimensions one through three take the same class of fixed smooth tests $F \perp e_g$ vanishing on the faces, and write $b_F = \sum_{k \neq k_g} b_k F_k$. If $b_F \neq 0$, its leading susceptibility is (1.9)*, with the square of this sum. If $b_F = 0$, its leading susceptibility is* (1.11)*, with*

$$\langle F, \mathcal{R}_\alpha F \rangle = \lim_{N \to \infty} \sum_{\substack{k,l \neq k_g \\ |k|_\infty, |l|_\infty \leq N}} F_k F_l \langle e_k, \mathcal{R}_\alpha e_l \rangle. \tag{1.21}$$

For each fixed k with $b_k = 0$, the Gaussian coefficient in (1.11) *is explicitly the spectral sum $\sum_{j \geq 2} \langle w_j, e_k \rangle^2 / (\mu_j - \mu_1)$, in any real orthonormal eigenbasis of H_α ordered so that $H_\alpha w_j = \mu_j w_j$ and $w_1 = w$. Equation* (1.15) *with $F = e_k$, $H = e_l$ gives the corresponding off-diagonal entries.*

Proof. The eigenpairs of H_1 are $(\pi^2|k|^2, e_k)$. Inserting them into (1.11) and (1.15) gives the four-dimensional assertions. For fractional powers, Parseval gives the stated expression for b_F and the spectral theorem gives the sum over w_j. Since $\|\mathcal{R}_\alpha\| = (\mu_2 - \mu_1)^{-1}$, rectangular sine truncations converge in its quadratic form for every $F \in L^2$. The susceptibility asymptotic for a smooth F is proved directly for its sampled source in Section 2. The infinite series in (1.20) and (1.21) therefore require only convergence in the bounded continuum form. □

A counterexample to the all-odd criterion at $d = d_c$. In four dimensions take $k_* = (3, 1, 1, 1)$. Integrating the sines gives $\int_\Omega e_{k_*} = \frac{1}{3} \int_\Omega e_g > 0$. Both sines thus couple to a uniform applied field. Their projections onto the non-Gaussian mode are $b_g = 1$ and $b_{k_*} = 0$, since $w = e_g$. The Dirichlet eigenvalue gap is $\pi^2(|k_*|^2 - |k_g|^2) = 8\pi^2$. Corollary 1.5 gives

$$\chi_{k_*,n} = \frac{n^2 A_n^{\mathrm{G}}}{8\pi^2} + o(n^2), \qquad \frac{\chi_{k_*,n}}{\chi_{k_g,n}} \asymp (\log n)^{-1/2} \longrightarrow 0. \tag{1.22}$$

Thus two indices in the all-odd column of Table I have different susceptibility scales at the upper critical dimension. This disproves the extension of its free-boundary Q assignment to $d = 2s$. Testing the power-law assignment in the table's stated range requires a mode estimate for $d > 2s$.

For $0 < \alpha < 1$, reflection symmetry still rules out enhancement whenever an index has an even entry. The fractional ground state has infinitely many nonzero higher sine coefficients, as proved in Appendix A.2. Choose two such indices k, l. The test $F = b_l e_k - b_k e_l$ is orthogonal to both e_g and w. Its susceptibility has order n^s, although the two diagonal contributions in (1.17) each have order $n^s/\sqrt{g}$. Their leading terms cancel with the mixed moment.

2 Proof of the susceptibility formulas

The origin of the two scales can be seen from a single field configuration $z = au$, with u a principal profile of order $V^{-1/2}$ per site. Its quartic energy has order ga^4/V, so the second moment of a has order $(V/g)^{1/2} = n^s/\sqrt{g}$. Projecting away that direction leaves a covariance of order n^s. To recover its coefficient for spins we follow a spatially varying magnetic source through the Gaussian integrations and retain the derivative of the quartic potential.

2.1 The principal profile and the reduced Green operator

Use the isometric cell embedding U_n that sends a lattice vector v to the function with value $n^{d/2}v_x$ on $\prod_i((x_i-1)/n, x_i/n)$, extended by zero outside Ω. Its adjoint takes normalized cell averages. Write $\mathcal{D}(u) = \mathcal{E}_\alpha(u,u)$ for the form in (1.6). The first estimate below controls the error in the overlap with w. A rate is needed because its squared overlap is multiplied by $n^s/\sqrt{g}$. The second estimate identifies the coefficient on the n^s scale.

Lemma 2.1 (Principal profile and reduced covariance)**.** *Take the prescribed Gaussian reference of [1, Section 2], with $B - A = \zeta J_n$, bulk row sum $p = \sum_z A_\infty(z) > 0$, and $G = a_R(A^{-1} - c'B^{-1})^{-1}$. Let ℓ be the positive solution of $A\ell = \eta_1 B\ell$ for the largest generalized eigenvalue $\eta_1 = \eta_{1,n}$, normalized by $\ell^{\mathsf{T}} B\ell = 1$. Put*

$$u_* = B\ell, \qquad G_h = G - (\ell^{\mathsf{T}} G\ell)u_* u_*^{\mathsf{T}}. \tag{2.1}$$

At fixed R, for $\delta = s/16$,

$$\|U_n u_* - \sqrt{p}\, w\|_{L^2} = O_R(n^{-\delta}). \tag{2.2}$$

For any lattice sequences v_n, z_n with $U_n v_n \to v$ and $U_n z_n \to z$ in $L^2(\Omega)$,

$$n^{-s}\langle v_n, G_h z_n\rangle \longrightarrow \frac{a_R p^2}{\zeta\kappa_R}\langle v, \mathcal{R}_\alpha z\rangle. \tag{2.3}$$

Proposition 2.7 of [1] controls the first generalized eigenvalue and the gap in the box. Appendix A compares the walk form with (1.6) to obtain the rate for the principal eigenvector and the limit of the reduced covariance. For smooth F the sampling satisfies $\|U_n f_n(F) - F\|_2 = O_F(n^{-1})$. Indeed the sample point and any point in its cell differ by $O(n^{-1})$, and the ratio of the two normalizing factors is $(n/(n+1))^{d/2} = 1 + O(n^{-1})$.

2.2 Smooth sources and the spin normalization

The microscopic split replaces a spin source f by a convolution of f restricted to the cube. For a signed source it is important to bound the boundary error on the n^s scale, even if the variance of the source itself is larger. Lemma 2.2 gives that bound.

Let ψ denote the full canonical field in the coordinates used at the final integration. Its Gaussian reference has covariance Σ. In the original coordinates the reference covariance is G of Lemma 2.1. The change of field is given by

$$\Sigma = tG + \varepsilon a_R A, \qquad S = G\Sigma^{-1}, \qquad h_{\mathrm{phys}} = a_R^{-1/2} S\psi. \tag{2.4}$$

For $d \le 3$ this gives $\Sigma = G$ and $S = I$. For $d = 4$ it is the change of variables of [1, Lemma 7.10] at the prescribed match. Write $\mathbb{E}_{\mathrm{full}}$ for expectation under the interacting law of ψ. For a covariance D, write $\mathbb{E}_D$ for expectation under the centred Gaussian of covariance D.

In the reference measure write $\psi = \psi_< + \psi_T$ as a sum of independent Gaussian fields of covariances $C_<$ and C_T. Thus $\Sigma = C_< + C_T$. The renormalisation group integrates $\psi_<$ first. Regressing ψ_T on its principal coordinate $a = \ell^{\mathsf{T}}\psi_T$ gives

$$\sigma_T^2 = \ell^{\mathsf{T}} C_T \ell, \qquad u = C_T\ell/\sigma_T^2, \qquad \psi_T = au + Y, \tag{2.5}$$

$$C_h = C_T - \sigma_T^2 uu^{\mathsf{T}}, \qquad K = C_< + C_h = \Sigma - \sigma_T^2 uu^{\mathsf{T}}. \tag{2.6}$$

Here a and Y are independent under the reference measure, with laws $N(0,\sigma_T^2)$ and $N(0,C_h)$. Thus u is the terminal regression profile and $u_* = B\ell$ is the profile for the full reference. In $d = 4$, a in (2.6) is the coordinate of the transformed field at cut k. The estimates below use its distribution at the temperature selected by a_0. The terminal test norm E_n in [1, Section 7.1] is the weighted difference norm specified in Appendix B.

Lemma 2.2 (Source directions and the return to spins)**.** *Let F, H be smooth on $\overline{\Omega}$ and vanish on $\partial\Omega$. Put $f = f_n(F) = (n+1)^{-d/2}F(\cdot/(n+1))$ and $h = f_n(H)$. At fixed d, R the source and covariance satisfy*

$$\|C_<\|_{2\to 2} + \|K\|_{2\to 2} = O_R(n^s), \qquad \|C_< f\|_{E_n} + \|Kf\|_{E_n} = O_{R,F}(1). \tag{2.7}$$

For the same tests, the covariance of the spins satisfies

$$\mathcal{C}_n(F,H) = \frac{1}{\beta a_R m^2}\mathbb{E}_{\text{full}}[\langle f,\psi\rangle\langle\psi,h\rangle] + o_{R,F,H}(n^s). \tag{2.8}$$

Set $c_{\text{sp}} = 1/(m\sqrt{\beta a_R})$. For every fixed $D < \infty$, uniformly for real $|\theta| \le Dn^{-s/2}$,

$$\mathbb{E}_{\text{sp}} e^{\theta S_n(f)} = (1+o(1))\,\mathbb{E}_{\text{full}} e^{\theta c_{\text{sp}} f^{\mathsf{T}}\psi}. \tag{2.9}$$

Proof. Appendix B proves the bounds on $C_<$ and K by summing the covariance slices with a smooth source. To transfer the covariance to spins, use the exact Hubbard–Stratonovich representation in physical units,

$$h_{\text{phys}} = a_R^{-1/2} S\psi = \sqrt{\beta}A\sigma + \eta_{\text{phys}}, \qquad \eta_{\text{phys}} \sim N(0,A) \tag{2.10}$$

in law, with independent noise. If $v = S^{-\mathsf{T}} f$ and $v_h = S^{-\mathsf{T}} h$, its covariance identity is

$$a_R^{-1}\mathbb{E}_{\text{full}}[\langle f,\psi\rangle\langle\psi,h\rangle] = \beta(Av)^{\mathsf{T}} G_{\text{sp}}(Av_h) + v^{\mathsf{T}} A v_h. \tag{2.11}$$

The noise term $v^{\mathsf{T}} A v_h$ is $O_{R,F,H}(1)$, since A and $S^{-\mathsf{T}}$ are uniformly bounded on ℓ^2 at fixed R.

For some $0 < \gamma < \min\{1, s\}$ the clipped rows satisfy $\|Af - pf\|_\infty = O_{R,F}(n^{-s-\gamma})$. Indeed the zero extension of F is Lipschitz, and the bulk kernel of A has a finite absolute γ-moment. For the fractional split the tail expansion has coefficients $O(j^{-1-s/2})$, and the j-step spread-out walk has γ-moment at most $C_R j^{\gamma/2}$. In dimension four the kernel has finite range. Transposing $S^{-1} = \Sigma G^{-1} = I - \varepsilon c' A B^{-1}$ gives $S^{-\mathsf{T}} = I - \varepsilon c' B^{-1} A$. Since B has bulk row sum p, finite range, and uniformly bounded inverse in absolute row norm, $Bf = pf + O_{R,F}(n^{-s-1})$ pointwise implies $B^{-1}f = p^{-1}f + O_{R,F}(n^{-s-1})$. The preceding bounds give

$$Av = mf + O_{R,F}(n^{-s-\gamma}) \quad \text{in supremum norm.} \tag{2.12}$$

The same identity in lower dimensions uses $m = p$.

Ferromagnetic positivity and the uniform susceptibility bound in [1, Lemma 7.5 and Theorem 1.1] give $G_{\text{sp}}(x,y) \ge 0$ and $\sum_{x,y}(G_{\text{sp}})_{xy} = O_R(Vn^s\sqrt{\log n})$. Consequently each error term in replacing

$(Av)^{\mathsf{T}} G_{\mathrm{sp}}(Av_h)$ by $m^2 f^{\mathsf{T}} G_{\mathrm{sp}} h$ is $O_{R,F,H}(n^{s-\gamma}\sqrt{\log n}) = o(n^s)$. The factors β and m are bounded above and away from zero at fixed R. This proves the covariance formula.

For exponential moments we need a bound on the source error for each spin configuration. Let $r = Av - mf$. In $d \le 3$ choose $s/2 < \gamma < \min\{1, s\}$ in the clipped-row estimate above. Since $V = n^{2s}$, we have $\|r\|_1 = O(n^{s-\gamma}) = o(n^{s/2})$. In $d = 4$ the kernels A and B are symmetric and have finite range at fixed R. In an interior row, Taylor expansion of F and cancellation of the linear term give errors $O_{R,F}(n^{-s-2})$ in $Af - pf$ and $Bf - pf$. In the fixed-width boundary collar, $F|_{\partial\Omega} = 0$ gives errors $O_{R,F}(n^{-s-1})$ on $O_R(n^{d-1})$ sites. Both ℓ^1 errors are therefore $O_{R,F}(n^{s-2}) = O_{R,F}(1)$. The identity $AB^{-1}Af - pf = (A - pI)f + AB^{-1}(A - B)f$ gives the same bound for r because A and B^{-1} have bounded absolute column sums. Thus $|\theta r^{\mathsf{T}}\sigma| \le |\theta|\|r\|_1 = o(1)$ at the stated scales in every dimension.

The independent Gaussian noise in the Hubbard–Stratonovich identity gives the exact relation

$$\mathbb{E}_{\mathrm{full}} e^{\theta c_{\mathrm{sp}} f^{\mathsf{T}}\psi} = \exp\left(\frac{\theta^2 v^{\mathsf{T}} A v}{2\beta m^2}\right) \mathbb{E}_{\mathrm{sp}} \exp\left(\frac{\theta}{m}(Av)^{\mathsf{T}}\sigma\right). \tag{2.13}$$

The first factor tends uniformly to one since $v^{\mathsf{T}} A v = O(1)$. The pointwise error estimate gives (2.9). □

2.3 Quartic moments of the principal coordinate

Let F_{in} be the density of the full interacting field relative to its Gaussian reference, up to a constant. Integration of the earlier scales gives $F_T(z) = \mathbb{E}_{C_<}[F_{\mathrm{in}}(z + \psi_<)]$, up to a positive field-independent factor. Thus the interacting law of ψ_T is the Gaussian law of covariance C_T weighted by $F_T(\psi_T)/Z$, where $Z = \mathbb{E}_{C_T}[F_T(\psi_T)]$. Write D_v for the directional derivative in the real field direction v. The principal coordinate has quartic scale $a_* = (V/g)^{1/4}$, corresponding to the site-field scale $h_* = a_*/\sqrt{V}$. With the normalized filter T described after Theorem 1.4, the terminal estimates give

$$\sigma_T^2 \asymp_R n^s/g, \qquad \sqrt{V}\,\|u\|_{E_n} + \|\ell\|_2 \le C_R, \qquad \mathbb{E}_{C_h} \exp(c_R n^s \|Y\|_{E_n}^2) \le C_R, \tag{2.14}$$

$$F_T(z) = \exp[-V_4(z)] + \mathcal{E}(z), \qquad V_4(z) = \frac{g}{4}\sum_x (Tz)_x^4. \tag{2.15}$$

The weight F_T is real and nonnegative. For $j = 0, 1, 2$ its remainder satisfies the Gaussian envelope

$$\mathbb{E}_{C_h}\left|D_{v_1}\cdots D_{v_j}\mathcal{E}(a_* y u + Y)\right| \le C_R \epsilon_n e^{-c_R y^2} \prod_{i=1}^{j} \frac{\|v_i\|_{E_n}}{h_*}, \qquad \epsilon_n \to 0. \tag{2.16}$$

Appendix B.2 specifies the matched-trajectory estimates that give (2.14)–(2.16) in each dimension. The same estimate for e^{-V_4} holds with ϵ_n replaced by one. In the proof of Proposition 7.4 of [1], the L^2 regulator gives the Gaussian decay in y because $\ell^{\mathsf{T}}(a_* y u + Y) = a_* y$. The remaining regulator is integrable by (2.14). For the pure quartic, the bounded ℓ^4 inverse of T gives $V_4(z) \ge c g \|z\|_2^4/V$. Its field derivatives give polynomials which are absorbed by a fraction of the quartic. These envelopes also permit any fixed polynomial weight in y under integration.

The coefficient of the enhanced susceptibility comes from the second moment of the scalar quartic density. Its fourth moment appears when the source differentiates the quartic and determines the correction from F to j_F in the mixed covariance.

Lemma 2.3 (Scalar moments and the differentiated quartic)**.** *Write $m_{2,n} = Z^{-1}\mathbb{E}_{C_T}[a^2 F_T(\psi_T)]$ for the second moment of the interacting principal coordinate. Define*

$$q_n = V\sum_x (Tu)_x^4, \qquad d_n = \frac{V}{q_n} T^{\mathsf{T}}[(Tu)^3], \tag{2.17}$$

where the cube in brackets is pointwise. Then

$$q_n \longrightarrow p^2 I_4, \qquad m_{2,n} = \frac{n^s}{\sqrt{g}} \frac{2\Gamma(3/4)}{\Gamma(1/4)\sqrt{q_n}}(1+o(1)), \tag{2.18}$$

$$\langle u, d_n\rangle = 1, \qquad U_n d_n \longrightarrow \frac{w^3}{\sqrt{p}\, I_4} \quad \text{in } L^2. \tag{2.19}$$

For real directions v with $\|v\|_{E_n} = O(1)$,

$$Z^{-1}\mathbb{E}_{C_T}[aD_vF_T] = -\langle d_n, v\rangle + o(n^s), \qquad Z^{-1}\mathbb{E}_{C_T}|D_v^2 F_T| = O_R(\sqrt{g}\, n^s). \tag{2.20}$$

The errors are uniform on bounded sets in the indicated norm.

Proof. The full covariance in the natural field is $\Sigma = tG + \varepsilon a_R A$. Because $A\ell = \eta_1 B\ell$ and $G\ell = (\ell^{\mathsf{T}} G\ell)u_*$, its full regression profile is exactly $u_* = B\ell$. Subtracting the preterminal column and its variance gives

$$u - u_* = \frac{(\ell^{\mathsf{T}} C_< \ell)u_* - C_<\ell}{\sigma_T^2}, \qquad \|u - u_*\|_2 = O_R(g) \tag{2.21}$$

by Lemma 2.2 and (2.14). Hence $U_n u \to \sqrt{p}\, w$ in L^2 by Lemma 2.1. The uniform bound on $U_n u$ also gives $w \in L^\infty$. At fixed R the normalized filter has bounded absolute row and column sums and microscopic range, with row sum one away from its boundary collar. The bulk filter is symmetric, and the same properties hold for its transpose. In $d \le 3$ the filter is the normalized finite Neumann sum in the proof of [1, Proposition 7.2]. In $d = 4$ it is the normalized short-walk filter of [1, Lemma 7.10]. On samples of a smooth compactly supported function these filters converge strongly to the identity by translation continuity or the first moment bound. Density and uniform operator bounds give the same assertion on any strongly convergent embedded sequence, including along the allowed counterterms. Thus $U_n Tu \to \sqrt{p}\, w$ strongly in L^2 and is uniformly bounded in L^∞. Taking the fourth power in L^1 and the cube in L^2, then applying T^{T} to the latter, gives the limits of q_n and $U_n d_n$. The identity $u^{\mathsf{T}} d_n = 1$ follows by contraction.

Write $a = a_* y$. On bounded y-sets, $g^{1/4}\|T(a_* yu)\|_4 = q_n^{1/4}|y|$, whereas $g^{1/4}\|TY\|_4 = O_{L^j}(g^{1/4})$ for every fixed j by (2.14) and $\|TY\|_4 \le CV^{1/4}\|Y\|_\infty$. Consequently $V_4(a_* yu + Y) - q_n y^4/4 \to 0$ in L^1, uniformly on these sets. The scalar Gaussian factor $\exp[-a_*^2 y^2/(2\sigma_T^2)]$ tends to one there. Equation (2.16) controls the tails in the integrals with weights $1, y^2, y^4$. In particular $Z \asymp_R a_*/\sigma_T$. After dividing by Z, the second and fourth moments of y equal those under the density proportional to $\exp(-q_n y^4/4)$, up to $o(1)$. The same comparison holds with e^{-V_4} in the numerator. The second moment of this scalar quartic law is $2\Gamma(3/4)/(\Gamma(1/4)\sqrt{q_n})$, and its fourth moment is $1/q_n$. This proves (2.18) and justifies moment convergence at the normalization used in Theorem 1.3.

For the first derivative of the pure quartic weight the insertion is $-ga\sum_x (T(au+Y))_x^3 (Tv)_x$. Replacing the cube by $a^3(Tu)_x^3$ gives a normalized error $o(n^s)$. In detail, a term with $j \ge 1$ factors of TY costs at most

$$Cg|a|^{4-j} V\, \|Tu\|_\infty^{3-j} \|TY\|_\infty^j \|Tv\|_\infty. \tag{2.22}$$

At $a = a_* y$ its Gaussian expectation, with the pure quartic weight, is bounded by $C_R n^s g^{j/4}(1 + |y|^4)e^{-c_R y^4}$. Here $V_4(au+Y) \ge c_R y^4$ follows by $\ell^{\mathsf{T}} Y = 0$ and the ℓ^4 inverse bound. The factor a_*/σ_T from integration cancels against Z. Since $ga_*^4 = V$, the retained term is

$$-\frac{V}{q_n}\sum_x (Tu)_x^3 (Tv)_x \frac{q_n}{Z}\mathbb{E}_{C_T}\Big[(a/a_*)^4 e^{-V_4(\psi_T)}\Big] = -\langle d_n, v\rangle + o(n^s). \tag{2.23}$$

Here the last factor tends to one by the quartic fourth moment, and $|\langle d_n, v\rangle| = O_R(\sqrt{V})$. The derivative of the remainder costs $O_R(\epsilon_n a_*/h_*) = o(n^s)$ by (2.16). Two derivatives cost $O_R(h_*^{-2}) = O_R(\sqrt{g}\, n^s)$ by the same envelope for F_T. This proves (2.20). □

2.4 Reconstruction of the spin covariance

Proof of the covariance formulas in Theorems 1.3 and 1.4. Set $f = f_n(F)$ and $b = u^{\mathsf{T}} f$. Exact Gaussian translation, first in the integrated field and then in the independent terminal variable Y, gives for $\eta \in \mathbb{R}$

$$\mathbb{E}_{\text{full}} e^{\eta f^{\mathsf{T}} \psi} = \frac{e^{\eta^2 f^{\mathsf{T}} K f/2}}{Z} \mathbb{E}_{C_T}\left[e^{\eta a b} F_T(au + Y + \eta K f)\right]. \tag{2.24}$$

The integrated field translates the interaction by $\eta C_< f$ and the terminal residual translates it by $\eta C_h f$. Gaussian translation holds on the support of Y even when C_h is singular. For fixed n and η near zero, a fraction of the Gaussian decay in the terminal bounds absorbs the translation and the linear source. This gives an integrable bound for the first two derivatives at zero.

Lemma 2.2 gives $\|Kf\|_{E_n} = O(1)$, so (2.20) applies with $v = Kf$. Two derivatives of (2.24) produce $f^{\mathsf{T}} K f$, $m_{2,n} b^2$, $2bZ^{-1}\mathbb{E}_{C_T}[aD_{Kf}F_T]$, and $Z^{-1}\mathbb{E}_{C_T}[D^2_{Kf}F_T]$. Completing the square in K yields

$$\mathbb{E}_{\text{full}}(f^{\mathsf{T}}\psi)^2 = L_n b^2 + (f - b d_n)^{\mathsf{T}} K (f - b d_n) + o(n^s), \qquad L_n = m_{2,n} - d_n^{\mathsf{T}} K d_n. \tag{2.25}$$

In fact the cross term from (2.24) is $2bZ^{-1}\mathbb{E}_{C_T}[aD_{Kf}F_T] = -2b\, d_n^{\mathsf{T}} K f + o(n^s)$. The bounded ℓ^2 norm of d_n gives $L_n = m_{2,n} + O_R(n^s)$. Applying (2.25) to $f + h$ and $f - h$ and taking one quarter of the difference gives the mixed moment for $f = f_n(F)$ and $h = f_n(H)$.

Put $f' = f - b d_n$, so that $u^{\mathsf{T}} f' = 0$ exactly. We can evaluate K on this test without differentiating in the direction $K d_n$. Indeed $K = \Sigma - \sigma_T^2 u u^{\mathsf{T}}$, and $u_*^{\mathsf{T}} f' = O_{R,F}(g)$ by the profile comparison in Lemma 2.3. The full Gaussian variance in the principal direction is $O_R(n^s/g)$. Since $A\ell = \eta_1 u_*$, removing that summand from Σ leaves exactly

$$t G_h + \varepsilon a_R (A - \eta_1 u_* u_*^{\mathsf{T}}). \tag{2.26}$$

The second term is bounded in operator norm. The omitted full principal term tested on f' costs $O_R(n^s g)$. The same estimates apply to bilinear tests. Equations (2.19) and Lemma 2.1 now give, with $h' = f_n(H) - (u^{\mathsf{T}} f_n(H)) d_n$,

$$U_n f' \to j_F, \qquad n^{-s}(f')^{\mathsf{T}} K h' = \frac{t a_R p^2}{\zeta \kappa_R}\, \mathcal{B}(F, H) + o(1). \tag{2.27}$$

The $o(1)$ holds for the actual bounded sequence $t = t_n$.

For the scalar term in (2.25), smooth sampling and the ground-state estimate give

$$u^{\mathsf{T}} f_n(F) = \sqrt{p}\, b_F + O_{R,F}(g + n^{-s/16}). \tag{2.28}$$

Multiplying this error by $L_n = O_R(n^s/\sqrt{g})$ still costs $o(n^s)$. Combining (2.25), (2.27), and Lemma 2.2 therefore gives the single bilinear formula

$$\mathcal{C}_n(F, H) = \frac{p L_n}{\beta a_R m^2}\, b_F b_H + n^s A_n^{\mathrm{G}} \mathcal{B}(F, H) + o(n^s). \tag{2.29}$$

Using (2.18) and $q_n \to p^2 I_4$ gives (1.9). When $b_F = 0$ the remaining form has $j_F = F$ and gives (1.11). Taking $F = H = e_g$ in (2.29) and eliminating its scalar coefficient gives (1.15). □

2.5 Exponential moments of the spin field

Proof of the exponential-moment assertions of Theorem 1.3. Set $f = f_n(F)$ and $b = u^{\mathsf{T}} f$. Fix $D > 0$ and take $\theta \in [-D, D]$. Write $\rho_n = g^{1/4} n^{-s/2}$ if $b_F \neq 0$ and $\rho_n = n^{-s/2}$ if $b_F = 0$. To compute $\mathbb{E}_{\mathrm{sp}} e^{\theta \rho_n X}$, use (2.9) and (2.24). The two factors are the Gaussian exponential $\exp(\eta_n^2 f^{\mathsf{T}} K f/2)$ and the translated terminal integral, with

$$\eta_n = \theta c_{\mathrm{sp}} \rho_n, \qquad v = \eta_n K f, \qquad \xi_n = \eta_n a_* b. \tag{2.30}$$

Lemma 2.2 gives $\|v\|_{E_n}/h_* = O(g^{1/4})$ at both scales. The profile comparison gives $b = \sqrt{p}\, b_F + O(g + n^{-s/16})$. Thus ξ_n stays bounded if $b_F \neq 0$. If $b_F = 0$, then $\xi_n = O_D(g^{-1/4}(g + n^{-s/16})) = o(1)$.

We first evaluate the terminal integral with $a = a_* y$. Set $\delta_n = \|v\|_{E_n}/h_*$. For $\phi = a_* y u + Y + v$, the identities $\ell^{\mathsf{T}} Y = 0$ and $\ell^{\mathsf{T}} u = 1$ imply $\|\phi\|_2/a_* \geq c(|y| - C\delta_n)_+$. The pointwise remainder bound used in Appendix B.2 therefore gives

$$\mathbb{E}_{C_h} |\mathcal{E}(a_* y u + Y + v)| \leq C \epsilon_n e^{-c y^2}, \qquad e^{-V_4(a_* y u + Y + v)} \leq \exp[-c(|y| - C\delta_n)_+^4]. \tag{2.31}$$

Indeed the positive exponent in the remainder regulator is bounded by $C c_h (y^2 + h_*^{-2} \|Y\|_{E_n}^2 + \delta_n^2)$, where c_h is the regulator constant of [1, Assumption 7.1]. Its y^2 term is absorbed by the negative quadratic exponent, as in the proof of (2.16), and (2.14) controls the expectation of the remaining factor. The quartic bound follows from $V_4(\phi) \geq c g \|\phi\|_2^4/V$.

On bounded y-intervals, $g^{1/4}\|Tv\|_4 \leq C\delta_n$ and $g^{1/4}\|TY\|_4 = O_{L^4}(g^{1/4})$. Hence $V_4(a_* y u + Y + v) = q_n y^4/4 + o_{L^1}(1)$ there, uniformly on each such interval. The scalar Gaussian factor tends to one there as well. Since $|\xi_n|$ is bounded, the bounds in (2.31) remain integrable after multiplication by $e^{\xi_n y}$. Dividing by the unshifted integral for Z cancels the Gaussian normalization $a_*/(\sqrt{2\pi}\sigma_T)$ and gives

$$\frac{\mathbb{E}_{C_T}\left[e^{\eta_n a b} F_T(a u + Y + v)\right]}{Z} = \frac{\int_{\mathbb{R}} e^{\xi_n y - q_n y^4/4}\, dy}{\int_{\mathbb{R}} e^{-q_n y^4/4}\, dy} + o(1) = \Phi_4(\xi_n q_n^{-1/4}) + o(1). \tag{2.32}$$

The denominators are bounded away from zero after cancellation since $q_n \to p^2 I_4 > 0$. All tail and compact-interval estimates are uniform for $|\theta| \leq D$.

If $b_F \neq 0$, then $\eta_n^2 f^{\mathsf{T}} K f = O(\sqrt{g})$. Also $\xi_n = \theta c_{\mathrm{sp}} b$, so that $\xi_n q_n^{-1/4} = \theta b_F \sqrt{A_n^{\mathrm{NG}}/v_4} + o(1)$ by (3.3). This proves (1.10). If $b_F = 0$, the terminal factor in (2.32) tends to one. Since $b = o(1)$ and $\|d_n\|_2 = O(1)$, replacing f by $f - b d_n$ in $n^{-s} f^{\mathsf{T}} K f$ costs $o(1)$ by Lemma 2.2. Now $j_F = F$, and (2.27) gives $\eta_n^2 f^{\mathsf{T}} K f = \theta^2 A_n^{\mathrm{G}} \langle F, \mathcal{R}_\alpha F \rangle + o(1)$. This proves (1.12). Finally, for $F \neq 0$ the susceptibility formulas give $\rho_n^2 \chi_n(F) \asymp_{R,F} 1$, since $\mathcal{R}_\alpha$ is strictly positive on $w^\perp$ and both amplitudes are bounded above and away from zero. For fixed $t \in \mathbb{R}$ take $\theta = t/(\rho_n \sqrt{\chi_n(F)})$. This choice stays in a bounded interval, and the two formulas give the Laplace transforms of the variance-one laws in (1.13). □

3 Spin amplitudes

In the microscopic split of [1, Section 2], A_∞ is the positive convolution kernel used for the Gaussian field and Q_∞ is the remaining finite-range spin interaction. Write A, Q for their restrictions to Λ_n, and put $B = I - Q$, retaining the diagonal of Q as in the Gaussian reference. The temperature and source normalizations are

$$B - A = \zeta J_n, \qquad \beta = \tau/\zeta, \qquad p = \sum_z A_\infty(z) = 1 - \sum_z Q_\infty(z) > 0. \tag{3.1}$$

Here $\zeta = (1+e^{-R})^{-1}$ for $d \le 3$, and $\zeta = (1+t_R)^{-1}$ for the local split with diagonal correction t_R in [1, Lemma 2.1]. Denote the microscopic field exponent a_* of [1, Section 2] by $\gamma_{\mathrm{sc}} \in (1/12, 1/10)$. Then $\lambda = R^{-2d\gamma_{\mathrm{sc}}}$ and $a_R = \lambda R^s$. The physical Hubbard–Stratonovich field in the original split equals the canonical field divided by $\sqrt{a_R}$.

Following Section 7.5 of [1], we write $c' = c'_n$ for the original reference parameter called c in its Section 2. It lies below the pole $c_{*,n} = \lambda_{\max}(B^{-1/2}AB^{-1/2})^{-1}$. In $d \le 3$ set $\varepsilon = 0$. In $d = 4$, let $\varepsilon = \varepsilon_n$ be the parameter of the change of field selected together with τ in [1, Corollary 7.13]. Put $t = 1-\varepsilon$ and $m = p(1-\varepsilon c')$. The coefficient g is normalized so that the terminal quartic is $(g/4)\sum_x (Tz)_x^4$, where the linear filter T has bulk row sum one. In $d \le 3$, $g = g_N$ is the terminal coupling of Proposition 7.14 of [1]. In $d = 4$ the prescribed coordinate a_0 at cut J fixes the temperature through its marginal density [1, Eq. (7.43)]. The proof of Corollary 7.13 then evaluates moments in the transformed field at the earlier cut $k = J-1$. We use $g = g_k$ there. The parameters t, m, β are bounded above and away from zero at fixed R. The amplitudes are

$$\sigma_R^2 = \sum_z z_1^2 D_R(z), \qquad \kappa_R = (\sigma_R^2/2)^\alpha, \tag{3.2}$$

$$A_n^{\mathrm{NG}} = \frac{2\Gamma(3/4)}{\Gamma(1/4)\sqrt{I_4}\,\beta a_R m^2}, \qquad A_n^{\mathrm{G}} = \frac{t}{\tau\,\kappa_R (1-\varepsilon c')^2}. \tag{3.3}$$

These are the amplitudes for the original ± 1 spins. For example, in $d \le 3$ the Gaussian amplitude is $1/(\tau\kappa_R)$. All statements allow the matched parameters to depend on n. A fixed numerical coefficient multiplying $n^s\sqrt{\log V}$ would require convergence of $A_n^{\mathrm{NG}}/\sqrt{g\log V}$, and a fixed Gaussian amplitude would require convergence of A_n^{G}.

A Spectral estimates on the cube

A.1 Convergence of the principal profile and the reduced resolvent

Proof of Lemma 2.1. The domain of (1.6) embeds compactly in $L^2(\Omega)$ by the translation criterion. For $s < 2$ the jump form has a strictly positive kernel between distinct points. Hence each real minimizer has one sign, since taking its absolute value strictly decreases the energy if both signs occur on sets of positive measure. Two nonzero orthogonal nonnegative minimizers would have disjoint supports and their difference would be a sign-changing minimizer. This proves simplicity. For $s = 2$ simplicity follows from the Dirichlet sine expansion.

Put $b = 1/n$ and $D_n(v) = b^{-s}\langle v, J_n v\rangle/\kappa_R$. In rescaled Fourier variables its zero-extension multiplier is $j_b(\xi) = b^{-s}(1-\widehat{D}_R(b\xi))^\alpha/\kappa_R$, where the walk multiplier is its characteristic function. On the Brillouin zone it is comparable at fixed R to $|\xi|^s$ since the support is aperiodic and the origin is a nondegenerate quadratic minimum of $1-\widehat{D}_R$. For all ξ, $j_b(\xi) \le |\xi|^s$ by $1-\cos t \le t^2/2$ and cube symmetry. On $|\xi| \le b^{-1/2}$, Taylor expansion gives $j_b(\xi) = |\xi|^s(1+O_R(b))$.

For $\|v\|_2 + D_n(v) \le C$, lattice translations of its zero extension obey $\|v(\cdot+m)-v\|_2 \le C_R|bm|^{s/2}$. This follows from $|e^{it}-1|^2 \le C|t|^s$ and the multiplier lower bound. For a translation of the embedded step function the squared norm is a convex combination of the squared norms for the adjacent lattice translations, by subdividing each cell. Consequently

$$\|U_n v(\cdot+y) - U_n v\|_2 \le C_R(|y|+b)^{s/2}, \qquad \|(B-pI)v\|_2 = O_R(b^{s/2}). \tag{A.1}$$

The second bound uses the finite-range convolution representing $Q = I - B$, with bulk row sum $1-p$, before restriction.

We give the spectral comparison including multiplicities. If z belongs to the continuum form domain, the difference-energy representation of J_n gives

$$D_n(U_n^* z) \le \int_{\mathbb{R}^d} j_b(\xi) |\widehat{z}(\xi)|^2 \, d\xi \le \mathcal{D}(z). \tag{A.2}$$

Indeed on the full cell grid averaging commutes with each lattice translation and is contractive, and all jump weights are nonnegative. The cell averages outside the cube are zero. Also $\|U_n U_n^* z - z\|_2 \le C b^{s/2} \mathcal{D}(z)^{1/2}$, by averaging the squared translation bound over pairs of points in each cell. Conversely, any unit sequence of bounded D_n has an embedded strongly convergent subsequence by the translation estimate and bounded support. Its limit z satisfies $\mathcal{D}(z) \le \liminf D_n(v)$. Indeed on each fixed Fourier ball the transform of $U_n v$ equals the normalized lattice transform times the cell multiplier, a phase times a product of sinc functions of modulus at most one. Integrate on this ball, use local convergence of j_b, and increase the radius.

Write $\lambda_{j,n}$ for the levels of the Rayleigh quotient $pD_n(v)/\langle v, Bv\rangle$. The sampled first j continuum levels give $\lambda_{j,n} \le \mu_j + O_{R,j}(b^{s/2})$. For the lower bound take a subsequence realizing the lower limit and extract strong limits of the first j vectors normalized in the B-inner product. Their multiples by $\sqrt{p}$ tend to an orthonormal family. Applying the form lower bound to every linear combination and then applying continuum min–max gives $\liminf \lambda_{j,n} \ge \mu_j$. The limits are eigenvectors at their respective levels. Indeed the ith limit has energy at most μ_i and is orthogonal to the previously obtained eigenvectors, so it attains the minimum on that complement. This argument allows arbitrary choices within repeated eigenspaces.

We need a rate only for the first vector. Put $v = \ell/\|\ell\|_2$ and $V_n = U_n v$. The upper comparison and the denominator estimate give $D_n(v) \le \mu_1 + O_R(b^{s/2})$. Let $r = b^{1/4}$ and smooth V_n by a nonnegative smooth probability density of support radius r, obtained by rescaling a fixed density. The result z is supported in the closure of $\Omega_r = (-r, 1+r)^d$ and $\|z - V_n\|_2 = O_R(r^{s/2})$. On $|\xi| \le b^{-1/2}$ the cell multiplier and the Fourier multiplier of this convolution each have modulus at most one. Thus this part of $\mathcal{D}(z)$ is at most $(1 + O_R(b))D_n(v)$. On the complement, rapid decay of the smooth multiplier gives $O(b^2)$ using $\|V_n\|_2 = 1$. The ground level and spectral gap on Ω_r are $(1+2r)^{-s}$ times their values on Ω. The Rayleigh excess of $z/\|z\|_2$ above this ground level is therefore $O_R(r^{s/2} + r + b)$. As $z \ge 0$, the gap implies that its distance to the normalized, dilated ground state w_r is $O_R((r^{s/2} + r + b)^{1/2})$. To bound $\|w_r - w\|_2$, smooth the compactly supported w at radius r with error $O(r^{s/2})$, and compare the dilates of the smooth function by integrating their derivatives in the dilation parameter. The cost is $O(r)(1 + \|\nabla(w * \rho_r)\|_2) = O(r^{s/2})$ by the Fourier form bound and bounded support. We obtain $\|V_n - w\|_2 = O_R(b^{s/16})$. Normalization gives $\|\ell\|_2 = p^{-1/2} + O_R(b^{s/2})$, and the estimate for $(B - pI)\ell$ proves the profile assertion.

For the reduced covariance, put $e_n = \zeta \kappa_R/(pn^s)$ and $r_n = (1 - 1/c')/e_n$. The eigenvalues of $E = B^{-1/2} A B^{-1/2}$ are $\eta_{j,n} = 1 - e_n \lambda_{j,n} > 0$. The pole displacement prescribed in [1, Section 2] is $c_{*,n} - c' = \theta_0 \lambda^2 \bar{g}_N (R/n)^s$, with fixed $\theta_0 > 0$ and $\bar{g}_N = O_R((\log n)^{-1})$. Hence $r_n = \lambda_{1,n} + o_R(1)$. If $\ell_{j,n}$ are the B-orthonormal generalized eigenvectors, the spectral decomposition of $G = a_R B^{1/2}(E^{-1} - c' I)^{-1} B^{1/2}$ gives

$$n^{-s} G_h = \frac{a_R p}{\zeta \kappa_R} \sum_{j \ge 2} \frac{\eta_{j,n}}{c'(\lambda_{j,n} - r_n)} (B\ell_{j,n})(B\ell_{j,n})^{\mathsf{T}}. \tag{A.3}$$

The operator norm of the tail from level m is bounded by $C_R/(\lambda_{m,n} - r_n)$, since $B^{1/2}\ell_{j,n}$ are orthonormal and B is uniformly bounded. Its upper limit tends to zero as $m \to \infty$. In each fixed group of complete eigenspaces the embedded columns tend subsequentially to $\sqrt{p}$ times an orthonormal continuum eigenbasis by the min–max comparison and the denominator estimate. The resulting spectral projections are independent of the chosen basis. Pairing with the strongly convergent tests and then removing the truncation gives the stated limit. □

A.2 Sine coefficients of the fractional ground state

We prove that w has infinitely many nonzero sine coefficients for $0 < \alpha < 1$. Suppose that w is a finite sine sum. Write T_t for the whole-space heat semigroup applied to its zero extension and restricted to Ω, and T_t^D for killed heat flow in Ω. The maximum principle gives $T_t w \geq T_t^D w$ for $w \geq 0$. The scalar identity $x^\alpha = c_\alpha \int_0^\infty (1 - e^{-tx}) t^{-1-\alpha}\, dt$ gives in the interior

$$(-\Delta_D)^\alpha w - H_\alpha w = c_\alpha \int_0^\infty (T_t w - T_t^D w) t^{-1-\alpha}\, dt. \tag{A.4}$$

One can first test this identity against nonnegative smooth functions of compact support. The integral over $1 \leq t \leq 2$ is bounded below near the centre of a face by a positive constant. Indeed $T_t w$ there has a positive lower bound by its Gaussian kernel, while the finite sine sum $T_t^D w$ vanishes uniformly at the face. On the left both the finite sine sum $(-\Delta_D)^\alpha w$ and $H_\alpha w = \mu_1 w$ vanish there. This contradiction rules out a finite sine expansion.

B Gaussian source bounds and terminal estimates

B.1 Covariance bounds for a smooth source

To prove the norm bounds in Lemma 2.2, let b be the length scale of a covariance slice and put

$$\rho_b(x) = \min\{b, R + \operatorname{dist}(x, \Lambda_n^c)\}, \qquad \|v\|_{b,\omega} = \max_{|\nu| \leq q} \sup_{x \in \Lambda_n} (b/\rho_b(x))^\omega \rho_b(x)^{|\nu|} |\nabla^\nu v(x)|. \tag{B.1}$$

Here ∇^ν is a lattice difference of multi-order ν on a stencil inside the box, and q is the fixed derivative order in the terminal field norm. Use the weight ω of the matched trajectory, with $\max\{0, (s-1)/2\} < \omega < \alpha$ and $\omega \neq 1/2$. In dimension one this weight is $\omega = 1/8$ [1, Proposition 7.14]. In every dimension the terminal norm E_n is the raw norm $F_{n,\omega}$ of [1, Section 7.1], denoted here by $\|\cdot\|_{n,\omega}$. Its order-zero term controls the supremum norm. Fix ω' with $\omega < \omega' < \alpha$.

Corollary 2.13 of [1] treats the integrated uniform source. For a signed smooth source we use its two covariance-leg estimates and keep the differences of the test function in the interior. First use the original split, with integrated covariance $C_<^0$. A slice C_j of range $O(b)$ has $O(b^d)$ sites per row. Applying Cauchy–Schwarz to the two covariance legs, each of size $O_R(b^{-s/2})$ at order zero, bounds the absolute row sum by $C_R b^s$. With k differences on the first leg, the slice comparison and the weighted stencil estimates of [1, Lemma 2.8, Proposition 2.9 and Lemmas 2.10–2.11] give, for $f = f_n(F)$ with $\|f\|_\infty \leq C_F n^{-s}$,

$$|\nabla^k (C_j f)(x)| \leq C_{R,F} n^{-s} b^s \rho_b(x)^{-k} (\rho_b(x)/b)^{\omega'}, \qquad 0 \leq k \leq q. \tag{B.2}$$

Within distance $O(b)$ of a face, $\rho_n \lesssim \rho_b$. Multiplying by the scale-n stencil weight bounds the last display by $C_{R,F} n^{-s+\omega} b^{s-\omega}$. In the interior the covariance row agrees with its bulk convolution, so the differences act on f. There $\|\nabla^k f\|_\infty \leq C_F n^{-s-k}$ on the support of the row and $b \lesssim \rho_n \leq n$, giving the same bound. The finitely many merged microscopic slices cost an R-dependent constant. Summing geometrically, using $\omega < s$ and $b \lesssim n$, gives $\|C_<^0 f\|_{E_n} = O_{R,F}(1)$. Summing the row bounds gives $\|C_<^0\|_{2\to 2} = O_R(n^s)$ by symmetry.

In dimension four use the unperturbed covariances $C_<^0, C_T^0$ at cut k. The transfer identity gives

$$C_< = tC_<^0 + (1-t) a_R A + t\eta M, \qquad C_T = t(C_T^0 - \eta M) \tag{B.3}$$

by [1, Lemma 5.4, Eq. (5.19)], applied at this cut in the proof of Corollary 7.13. Here η is bounded, $0 \leq M \leq a_R A$, and $M \leq C_R C_T^0$ for the original terminal covariance C_T^0 at this cut. Both extra

terms have bounded operator norm. The finite-range kernel A acts on f by bulk convolution outside a microscopic collar. Inside it the weighted differences cost $O_{R,F}(n^{\omega-s})$. For any weighted stencil D in $\|\cdot\|_{n,\omega}$, the conditioned-field and profile bounds at the earlier cut in [1, proof of Corollary 7.13] give

$$DC_T^0D^{\mathsf{T}} \le C_R(n^{-2}+\sigma_0^2n^{-4}), \qquad \sigma_0^2=\ell^{\mathsf{T}}C_T^0\ell = O_R(n^2\log n). \tag{B.4}$$

Thus $|DMf| \le (DMD^{\mathsf{T}})^{1/2}(f^{\mathsf{T}}Mf)^{1/2} \le C_{R,F}(n^{-1}+\sigma_0 n^{-2})$. This proves both estimates for $C_<$ under the transfer.

Finally (2.14) gives $\mathbb{E}_{C_h}\|Y\|_{E_n}^2 = O_R(n^{-s})$, hence $\|C_h\|_{2\to 2} \le \mathbb{E}_{C_h}\|Y\|_2^2 = O_R(Vn^{-s}) = O_R(n^s)$. In $C_hf = \mathbb{E}_{C_h}[Y\langle Y,f\rangle]$, Cauchy–Schwarz gives $\|C_hf\|_{E_n} \le (\mathbb{E}_{C_h}\|Y\|_{E_n}^2)^{1/2}(f^{\mathsf{T}}C_hf)^{1/2} = O_{R,F}(1)$. Adding the estimates for $C_<$ gives the bounds for K in (2.6).

B.2 Terminal estimates at the matched point

For $d=1,2,3$, Proposition 7.14 of [1] verifies Assumption 7.1 there at the physical temperature used in Theorem 1.3. It gives $g=g_N \asymp_R (\log V)^{-1}$, the variance, profile and conditioned Gaussian moment in (2.14), and the pointwise bound on the remainder and its first two derivatives in arbitrary field directions. In particular, the terminal packing estimate [1, Proposition 7.2] gives

$$\epsilon_n \le C_R(\sqrt{g}+g^{-9/4}\|K_N^{\mathrm{RG}}\|) \le C_R(\sqrt{g}+g^{3/4}+Ud_0^Ng^{-9/4}) = o(1). \tag{B.5}$$

Here K_N^{RG} is the terminal polymer activity, $U=O_R(1)$ is the initial activity bound, $0<d_0<1$ and $N \asymp \log n$. Integrating the pointwise bound over the conditioned field Y, as in the proof of [1, Proposition 7.4], gives (2.16) with the same factors $\|v_i\|_{E_n}/h_*$.

For $d=4$, use the temperature and change of field in [1, Corollary 7.13]. Its proof uses the prescribed scalar density at cut J to select the mass. At the earlier cut k, the terminal coefficients of the transformed field satisfy the bounds of [1, Proposition 7.2] at that mass, with $g=g_k \asymp_R (\log n)^{-1}$. Lemma 5.4 of [1], applied at the fixed change of aspect as in the proof of Corollary 7.13, gives $\sigma_T^2 \asymp_R n^2/g_k$, the profile bound for u and the exponential moment of Y in (2.14). The same integration of the two-derivative bound gives (2.16) for the field in (2.6). Thus the moment calculation uses only the coefficient estimates at the selected mass.

C Displacement of the pseudocritical point

Recall that $\tau_c=\tau_c(d,R)=\inf\{\tau>0:\langle\sigma_0\rangle_\tau^+>0\}$ and $\Delta_n=\tau_{\Lambda_n}^{\mathrm{pc}}-\tau_c$ and $(u)_+=\max\{u,0\}$.

Proof of Theorem 1.2. Let $G_{n,\tau}^{\mathrm{F}}$ be the zero-field free-box two-point matrix at physical inverse temperature τ and put $\chi_n^{\mathrm{unif}}(\tau)=V^{-1}\mathbf{1}^{\mathsf{T}}G_{n,\tau}^{\mathrm{F}}\mathbf{1}$. At the matched point $G_{n,\tau_{\Lambda_n}^{\mathrm{pc}}}^{\mathrm{F}}=G_{\mathrm{sp}}$ and [1, Theorem 1.1] gives

$$\chi_n^{\mathrm{unif}}(\tau_{\Lambda_n}^{\mathrm{pc}}) \asymp_{d,R} n^s\sqrt{\log n}. \tag{C.1}$$

The ferromagnetic pair kernel is $K_R(z)=-(L_R^\alpha)(0,z)$ for $z\neq 0$, with $K_R(0)=0$. Thus $K_R(x-y)=-(J_n)_{xy}$ at distinct sites in the box. When $\alpha<1$ the binomial series gives

$$I-L_R^\alpha=\sum_{k\ge 1}a_kD_R^{*k}, \qquad a_k=(-1)^{k+1}\binom{\alpha}{k}>0, \qquad \sum_{k\ge 1}a_k=1. \tag{C.2}$$

At $\alpha=1$, $K_R=D_R$. Hence $j_R=\sum_z K_R(z)$ lies in $(0,1]$. For $\tau j_R<1$ the Dobrushin influence sum is at most $\sum_z \tanh(\tau K_R(z)) \le \tau j_R<1$. Uniqueness and spin-flip symmetry imply $\tau_c \ge j_R^{-1} \ge 1$. The

matching construction in [1, Corollaries 6.3 and 7.13, Proposition 7.14] takes place in $|\beta-1| \le A_d R^{-d}$, with A_d fixed before R and n. Since $\tau = \zeta\beta$ and $0 < \zeta \le 1$ in Section 1, this gives $\tau^{\mathrm{pc}}_{\Lambda_n} \le 1 + A_d R^{-d}$ and hence the stated R^{-d} bound in all four dimensions.

Set $h_x(\tau) = \sum_{y\in\Lambda_n} G^{\mathrm{F}}_{n,\tau}(x,y)$ and $H_n(\tau) = \max_{x\in\Lambda_n} h_x(\tau)$. The bulk input is the free-state estimate $\langle\sigma_0\sigma_z\rangle^{\mathrm{f}}_{\tau_c} \le C_{d,R}(1+|z|_\infty)^{-s}$ of [16, Theorem 1]. By Griffiths' domain monotonicity the free-box correlations at τ_c satisfy the same bound. Summing over shells about x gives

$$H_n(\tau_c) \le C_{d,R} \sum_{k=0}^{n-1} (1+k)^{d-1-s} \le A_{d,R} n^s, \tag{C.3}$$

since $d - s = s$. At the matched point the maximum row sum dominates its average, so (C.1) gives $H_n(\tau^{\mathrm{pc}}_{\Lambda_n}) \ge a_{d,R} n^s \sqrt{\log n}$. For large n this exceeds $H_n(\tau_c)$. Monotonicity in τ then gives $\tau^{\mathrm{pc}}_{\Lambda_n} > \tau_c$.

For brevity write $G = G^{\mathrm{F}}_{n,\tau}$. Differentiating the spin expectation with respect to τ and applying Lebowitz's zero-field four-point inequality [17] gives

$$\frac{d}{d\tau} G(x,y) = \sum_{\{u,v\}\subset\Lambda_n} K_R(u-v)\,\mathrm{Cov}^{\mathrm{F}}_{n,\tau}(\sigma_x\sigma_y, \sigma_u\sigma_v) \le \sum_{u,v\in\Lambda_n} G(x,u) K_R(u-v) G(v,y). \tag{C.4}$$

The two cross-pairings turn the unordered bond sum into the ordered sum on the right. Summing over y yields $h'_x(\tau) \le j_R h_x(\tau) H_n(\tau)$. The maximum of these finitely many smooth row sums is locally Lipschitz and satisfies $H'_n \le j_R H_n^2$ almost everywhere. Integrating the differential inequality for $1/H_n$ gives

$$\Delta_n \ge \frac{1}{j_R}\left(\frac{1}{H_n(\tau_c)} - \frac{1}{H_n(\tau^{\mathrm{pc}}_{\Lambda_n})}\right) \ge \frac{n^{-s}}{j_R}\left(\frac{1}{A_{d,R}} - \frac{1}{a_{d,R}\sqrt{\log n}}\right). \tag{C.5}$$

The last parenthesis is bounded below by a positive constant for sufficiently large n.

For (1.2) take $d < 4$. If $\Delta_n \le 0$ the bound is immediate. Otherwise set $\tau = \tau^{\mathrm{pc}}_{\Lambda_n}$, $\theta = (\tau_c + \tau)/2$, and let D_r be the uniform kernel of Section 1 with integer spread $1 \le r \le n$. Two estimates on K_R will be useful. For a nonnegative symmetric coupling H with zero diagonal, denote its missing bond weight per site by $e_n(H) = V^{-1}\sum_{x\in\Lambda_n, y\notin\Lambda_n} H(x-y)$. At fixed R,

$$\min_{0<|z|_\infty\le r} K_R(z) \ge c_{d,R} r^{-d-s}, \qquad e_n(K_R) \le C_{d,R}\begin{cases} n^{-s}, & s<1,\\ n^{-1}\log(2n), & s=1,\\ n^{-1}, & s>1. \end{cases} \tag{C.6}$$

For the first estimate, $a_k \asymp_d k^{-1-s/2}$ in (C.2), and the local central limit theorem for the aperiodic spread-out walk gives $D^{*k}_R(z) \ge c_{d,R} k^{-d/2}$ for $r^2 \le k \le 2r^2$ and $|z|_\infty \le r$ and large r. Positivity in (C.2) covers the finitely many smaller r. For the second estimate, a displacement z leaves the cube from at most a fraction $\min\{1, d|z|_\infty/n\}$ of its sites. The walk after k steps has expected norm at most $C_{d,R}\sqrt{k}$. Splitting $\sum_{k\ge1} a_k \min\{1, C_{d,R}\sqrt{k}/n\}$ at $k = n^2$ gives the three bounds on $e_n(K_R)$.

Choose $\delta = c_{d,R}(\tau-\theta) r^{-s}$ with the constant small enough that $H_0 = \theta K_R$ and $H_1 = H_0 + \delta D_r$ satisfy $H_0 \le H_1 \le \tau K_R$ entrywise. Let $P_n(H)$ and $P_\infty(H)$ be the free-box and bulk log partition functions per site with pair coupling H. Tiling a large cube by n-boxes gives $0 \le P_\infty(H) - P_n(H) \le e_n(H)/2$, since adding the bonds between tiles increases the pressure by at most their total weight per site. Interpolation between H_0 and H_1, Griffiths monotonicity and $\|D_r\|_\infty \le C_d r^{-d}$ give

$$P_n(H_1) - P_n(H_0) \le \frac{\delta}{2V} \sum_{x,y\in\Lambda_n} D_r(x-y) G^{\mathrm{F}}_{n,\tau}(x,y) \le C_d \delta r^{-d} \chi^{\mathrm{unif}}_n(\tau). \tag{C.7}$$

Write $m_+(\theta) = \langle\sigma_0\rangle_\theta^+$. In the bulk, $P_\infty(H_1) - P_\infty(H_0) \geq \delta m_+(\theta)^2/2$. Indeed, keep n, r fixed and use the same H_0 plus boundary field in both partition functions on an auxiliary M-box. Positive association and plus-volume monotonicity bound every H_0 pair expectation below by the squared plus magnetization. Jensen's inequality for the added interior bonds gives the claim as $M \to \infty$, since D_r has total mass one and the boundary weight per site tends to zero by summability. Combining the pressure estimates and using $e_n(H_1) \leq \tau e_n(K_R)$ yields

$$0 < m_+(\theta)^2 \leq C_d r^{-d} \chi_n^{\mathrm{unif}}(\tau) + \tau e_n(K_R)/\delta. \tag{C.8}$$

By the magnetization inequality [18, Theorem 2.1(1)],

$$m_+(\theta)^2 \geq 1 - (\tau_c/\theta)^2 = \frac{\Delta_n(\theta + \tau_c)}{2\theta^2} \geq \frac{\Delta_n}{2\theta}. \tag{C.9}$$

Here $\theta \leq \tau \leq 1 + A_d R^{-d}$ and $\delta = c_{d,R} \Delta_n r^{-s}/2$. Multiplying the resulting inequality by Δ_n and taking the positive quadratic root gives

$$(\Delta_n)_+ \leq C_{d,R} \left(\frac{\chi_n^{\mathrm{unif}}(\tau_{\Lambda_n}^{\mathrm{pc}})}{r^d} + \sqrt{e_n(K_R) r^s} \right). \tag{C.10}$$

Using (C.1), choose r to balance the two terms. Up to integer rounding, the choices are $n^{3/5}(\log n)^{2/5}$ for $d = 1$, $n^{3/5}$ for $d = 2$, and $n^{8/15}(\log n)^{2/15}$ for $d = 3$. Substitution gives (1.2).

To conclude convergence, for each fixed $0 < \tau < \tau_c$, subcritical sharpness [18, Theorem 2.1(2)] and domain monotonicity bound $\chi_n^{\mathrm{unif}}(\tau)$ by the finite bulk susceptibility. By (C.1) and monotonicity, $\tau_{\Lambda_n}^{\mathrm{pc}} > \tau$ eventually. This gives $\liminf_n \tau_{\Lambda_n}^{\mathrm{pc}} \geq \tau_c$ in every dimension. For $d < 4$, (1.2) supplies the corresponding limsup.

For $d = 4$, $K_R = D_R$ has finite range and contains every nearest-neighbour bond. At any fixed $\tau > \tau_c$, slab percolation and equality of the free and wired bulk FK measures hold [19, Sections 2.1–2.3]. These are the hypotheses of the finite-range FK–Ising crossing-cluster estimate [20, Section 3.2, Eq. (3.6)]. Take a central block of dyadic side between $n/8$ and $n/4$. For large n all bonds touching the block lie in Λ_n. The estimate is uniform over the boundary wiring induced by bonds elsewhere in the free cube. Thus, with probability at least $1/2$, a cluster of the free cube contains at least $b_{\tau,R} V$ sites for some $b_{\tau,R} > 0$. The Edwards–Sokal identity gives $\chi_n^{\mathrm{unif}}(\tau) = V^{-1} \mathbb{E}_{\mathrm{FK}} \sum_{\mathcal{C}} |\mathcal{C}|^2 \geq b_{\tau,R}^2 V/2$, the sum running over the clusters of the free cube. Since (C.1) gives $\chi_n^{\mathrm{unif}}(\tau_{\Lambda_n}^{\mathrm{pc}}) = O_R(n^2\sqrt{\log n}) = o(V)$, monotonicity puts $\tau_{\Lambda_n}^{\mathrm{pc}} < \tau$ eventually for every fixed $\tau > \tau_c$. This proves convergence in dimension four. □

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