2D Discrete Gaussian model at high temperature

- The infinite-volume scaling limit -

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1st March, 2022

Based on the works with Roland Bauerschmidt and Pierre-François Rodriguez

- The Discrete Gaussian model, I. Renormalisation group flow at high temperature, arXiv:2202.02286 (2022)
- The Discrete Gaussian model, II. Infinite-volume scaling limit at high temperature, arXiv:2202.02287 (2022)

The Discrete Gaussian (dgauss) model

- $\Lambda_N = 2$ dimensional discrete torus of side length L^N, distinguished point 0.
- $\Omega_N = \{\sigma \in (2\pi\mathbb{Z})^{\Lambda_N} : \sigma_0 = 0\}$.
- **Hamiltonian** $H_N^{\mathrm{DG}}(\sigma) = \frac{1}{4}\sum_{x\sim y\in\Lambda_N}(\sigma_x - \sigma_y)^2$ and

$$\mathbb{P}^{\mathrm{DG}}_{\Lambda_N,\beta}(\{\sigma\}) = \exp\Big(-\frac{1}{\beta}H_N^{\mathrm{DG}}(\sigma)\Big) \;/\; Z^{\mathrm{DG}}_{\Lambda_N,\beta}$$

  where

$$Z^{\mathrm{DG}}_{\Lambda_N,\beta} = \sum_{\sigma\in\Omega_N} e^{-\frac{1}{\beta}H_N^{\mathrm{DG}}(\sigma)}.$$

- Also called $\mathbb{Z}$-Ferromagnet or Integer-Valued GFF.

The Discrete Gaussian (dgauss) model

- Hamiltonian $H_N^{\mathsf{DG}}(\sigma) = \frac{1}{4}\sum_{x\sim y\in\Lambda_N}(\sigma_x-\sigma_y)^2$ and

$$\mathbb{P}^{\mathsf{DG}}_{\Lambda_N,\beta}(\{\sigma\}) = \exp\Big(-\frac{1}{\beta}H_N^{\mathsf{DG}}(\sigma)\Big)\ /\ Z^{\mathsf{DG}}_{\Lambda_N,\beta}.$$

- The dgauss with **spread-out interaction** with range $\rho \geq 1$:

$$H_N^{\mathsf{DG},\rho}(\sigma) = \frac{1}{2((2\rho+1)^2-1)}\sum_{|x-y|_\infty\leq\rho}(\sigma_x-\sigma_y)^2$$

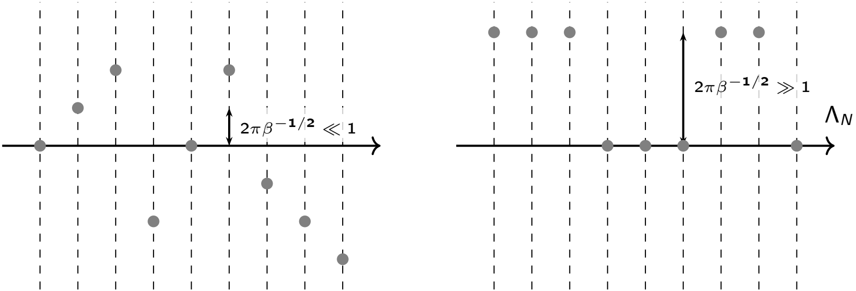
$$\mathbb{P}^{\mathsf{DG},\rho}_{\Lambda_N,\beta}(\sigma) \propto \exp\Big(-\frac{1}{\beta}H_N^{\mathsf{DG},\rho}(\sigma)\Big).$$

The Discrete Gaussian (dgauss) model

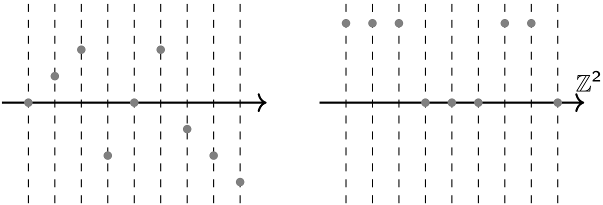
$$\mathbb{P}^{\mathrm{DG}}_{\Lambda_N,\beta}(\{\sigma\}) = \frac{\exp\left(-\frac{1}{\beta}H_N^{\mathrm{DG}}(\sigma)\right)}{Z^{\mathrm{DG}}_{\Lambda_N,\beta}}, \quad Z^{\mathrm{DG}}_{\Lambda_N,\beta} = \sum_{\sigma\in\Omega_N} e^{-\frac{1}{\beta}H_N^{\mathrm{DG}}(\sigma)}.$$

Since $\beta^{-1}H_N^{\mathrm{DG}}(\sigma) = H_N^{\mathrm{DG}}(\beta^{-1/2}\sigma)$, let $\Omega_{N,\beta} = \beta^{-1/2}\Omega_N$, then

$$Z^{\mathrm{DG}}_{\Lambda_N,\beta} = \sum_{\sigma\in\Omega_{N,\beta}} e^{-H_N^{\mathrm{DG}}(\sigma)}$$



Two phases of dgauss model



"Delocalisation phase" "Localisation phase"

Kosterlitz-Thouless phase transition

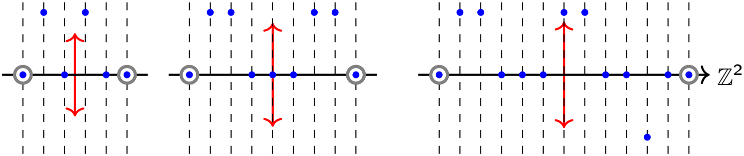
- (2D) XY model, Villain model, Coulomb gas, Sine-Gordon model

Two phases of dgauss model

Comparison of the Gibbs states

Investigation of the case of the 0-Dirichlet boundary condition :

- Localisation phase



bounded variance + strong clustering [BW'82]

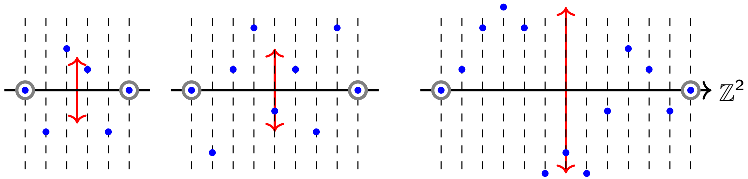
$\Rightarrow$ translation invariant Gibbs measure with slope 0

Two phases of dgauss model

Comparison of the Gibbs states

Investigation of the case of the 0-Dirichlet boundary condition :

- Delocalisation phase



unbounded variance

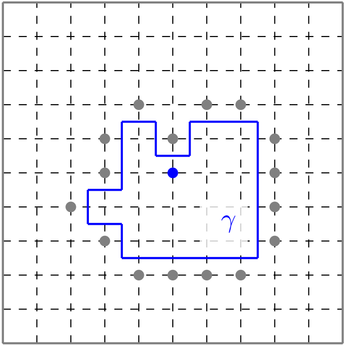
+ non-existence of transl. inv. Gibbs measure with slope 0 [S'06]

$\Rightarrow$ has to consider gradient Gibbs states

Two phases of dgauss model

Localisation phase, $\beta \ll 1$

- Peierls' argument $\mathrm{Var}_{\Lambda_N,\beta}(\sigma_x) \leq C$.
- [Brandenberger-Wayne(1982)] Strong clustering property.
- [Lubetzky-Martinelli-Sly(2014)] Extremal process.



$$\mathbb{P}(\sigma_0 \geq 1) \leq \sum_{n \geq 4} \sum_{\gamma : |\gamma| = n} \mathbb{P}(\gamma)$$

$$\to 0 \text{ as } \beta \to 0.$$

● $\sigma_x \leq 0$ ● $\sigma_x \geq 1$

Two phases of dgauss model

Localisation phase, $\beta \ll 1$

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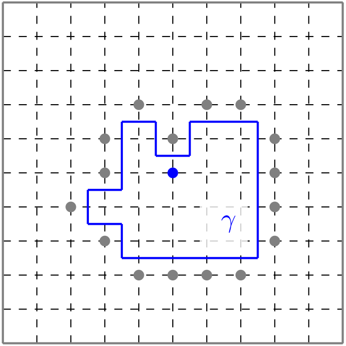

$$\mathbb{P}(\sigma_0 \ge 1) \le \sum_{n\ge 4} 4^n e^{-\beta^{-1} n}$$

$$\le Ce^{-\beta^{-1}} \text{ as } \beta \to 0.$$

$\bullet$ $\sigma_x \le 0$ $\bullet$ $\sigma_x \ge 1$

Two phases of dgauss model

Localisation phase, $\beta \ll 1$

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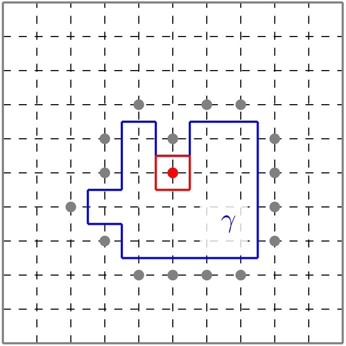


● $\sigma_x \leq 0$ ● $\sigma_x \geq 1$ ● $\sigma_x \geq m$

$$\mathbb{P}(\sigma_0 \geq m) \leq (Ce^{-\beta^{-1}})^m$$

$$\Rightarrow \quad \mathrm{Var}(\sigma_0) \leq Ce^{-\beta^{-1}}$$

Two phases of dgauss model

Delocalisation phase, $\beta \gg 1$

$f : \Lambda_N \to \mathbb{R}$ such that $\sum_{x \in \Lambda_N} f(x) = 0$,

- [Fröhlich-Park(1977)]

$$\mathbb{E}^{\mathrm{DG}}_{\Lambda_N,\beta}\left[e^{(f,\sigma)}\right] \leq \mathbb{E}^{\mathrm{GFF}}_{\Lambda_N,\beta}\left[e^{(f,\sigma)}\right] = e^{\frac{\beta}{2}(f,(-\Delta)^{-1}f)}$$

- [Fröhlich-Spencer(1981)] There is $\epsilon_1(\beta) = o(1)$ (as $\beta \to \infty$) such that,

$$e^{\frac{\beta(1+\epsilon_1(\beta))}{2}(f,(-\Delta)^{-1}f)} = \mathbb{E}^{\mathrm{GFF}}_{\Lambda_N,\beta(1+\epsilon_1(\beta))}\left[e^{(f,\sigma)}\right] \leq \mathbb{E}^{\mathrm{DG}}_{\Lambda_N,\beta}\left[e^{(f,\sigma)}\right].$$

- [Garban-Sepúlveda(2021)] With 0-Dirichlet boundary, if f has bounded support,

$$\mathrm{Var}^{\mathrm{DG}}_{\Lambda_N,\beta}[(f,\sigma)] \leq \left(1 - 2\beta e^{-\frac{\beta}{2}+o(\beta)}\right) \mathrm{Var}^{\mathrm{GFF}}_{\Lambda_N,\beta}[(f,\sigma)].$$

Two phases of dgauss model

Delocalisation phase, $\beta \gg 1$

- [Sheffield(2006)] Uniqueness of gradient Gibbs state.
- [Lammers(2021)] Existence of delocalisation phase for planar graphs of degree ≤ 3.
- [Aizenman-Harel-Peled-Shapiro(2021)] Existence of delocalisation phase for doubly periodic graphs.
- [Lammers-Ott(2021)] Discussion on dichotomy of the phases.

Delocalisation phase of dgauss model

Theorem 1 [II, B.-P.-R., 2022]

"Scaling limit of dgauss is a Gaussian free field on $\mathbb{R}^2$"

Let $\beta > 0$ **fixed** large, $f \in C_c^\infty(\mathbb{R}^2)$ be such that $\int_{\mathbb{R}^2} f = 0$, and for each $\epsilon > 0$, let $f_\epsilon(x)$ be the discretisation of $\epsilon^{-1} f(\epsilon x)$ on $\mathbb{Z}^2$. Then

$$\lim_{\epsilon \to 0} \log \mathbb{E}^{\mathrm{DG}}_{\mathbb{Z}^2,\beta}[e^{(f_\epsilon,\sigma)_{\mathbb{Z}^2}}] = \frac{\beta(1+\mathsf{s}^{\mathrm{DG}}(\beta))^{-1}}{2}(f,(-\Delta_{\mathbb{R}^2})^{-1}f)_{\mathbb{R}^2}$$

where $\mathsf{s}^{\mathrm{DG}}(\beta) = O(e^{-c\beta})$, $c > 0$.

Delocalisation phase of dgauss model

Theorem 2 [I, B.-P.-R., 2022]

"Scaling limit of dgauss is a Gaussian free field on $\mathbb{T}^2$"

For $\beta > 0$ sufficiently large, $f \in C^\infty(\mathbb{T}^2)$ be such that $\int_{\mathbb{T}^2} f = 0$, and for each $N > 0$, $f_N(x)$ be the discretisation of f on Λ_N. Then

$$\lim_{N\to\infty} \log \mathbb{E}^{\mathrm{DG}}_{\Lambda_N,\beta}[e^{(f_N,\sigma)_{\mathbb{T}^2}}] = \frac{\beta(1+\mathsf{s}^{\mathrm{DG}}(\beta))^{-1}}{2}(f,(-\Delta_{\mathbb{T}^2})^{-1}f)_{\mathbb{T}^2}$$

with the same $\mathsf{s}^{\mathrm{DG}}(\beta) = O(e^{-c\beta})$ as in Theorem 1.

Delocalisation phase of dgauss model

Recall : the **spread-out interaction**

$$H_N^{\mathrm{DG},\rho}(\sigma) = \frac{1}{2((2\rho+1)^2-1)} \sum_{|x-y|_\infty \leq \rho} (\sigma_x - \sigma_y)^2$$

Theorem 3 [I, B.-P.-R., 2022]

Let $\beta > 0$ sufficiently large, $f \in C^\infty(\mathbb{T}^2)$ be such that $\int_{\mathbb{T}^2} f = 0$, and for each $N > 0$, $f_N(x)$ be the discretisation of f on Λ_N. Then

$$\lim_{N\to\infty} \log \mathbb{E}_{\Lambda_N,\beta}^{\mathrm{DG},\rho}[e^{(f_N,\sigma)_{\mathbb{T}^2}}] = \frac{\beta(1+\mathsf{s}^{\mathrm{DG}}(\beta,\rho))^{-1}}{2v_\rho^2}(f,(-\Delta_{\mathbb{T}^2})^{-1}f)_{\mathbb{T}^2}$$

with $\mathsf{s}^{\mathrm{DG}}(\beta,\rho) = O(e^{-c\beta})$.

Open questions

Critical phenomena

1. GFF as a scaling limit : when $\beta = \beta_c$,

$$\lim_{\epsilon \to 0} \log \mathbb{E}^{\mathsf{DG}}_{\mathbb{Z}^2,\beta}[e^{(f_\epsilon,\sigma)}] \to \frac{\beta(1+\mathsf{s}^{\mathsf{DG}}(\beta))^{-1}}{2}(f,(-\Delta)^{-1}f)_{\mathbb{R}^2}$$

: within reach for spread-out interaction models.

2. The behaviour of s^{DG} as $\beta \downarrow \beta_c$.

3. Decay of the correlation function when $\beta \uparrow \beta_c$.

Open questions

Observables

1. Two-point function.
2. Convergence of discrete observables to an SLE.

Related models

1. Villain model
   - Two-point observables translate into line observables.
   - Non-trivial co-homology of torus is an obstacle.
2. Non-quadratic potentials (SOS model, dual of XY model) :

$$\mathbb{P}_{\Lambda_N,\beta}(\sigma) \propto e^{-\beta^{-1}\sum_{x\sim y} V(\sigma_x-\sigma_y)}.$$

Delocalisation phase of dgauss model

Theorem 2 [I, B.-P.-R., 2022]

"Scaling limit of dgauss is a Gaussian free field on $\mathbb{T}^2$"
Let $\beta > 0$ sufficiently large, $f \in C^\infty(\mathbb{T}^2)$ be such that $\int_{\mathbb{T}^2} f = 0$, and for each $N > 0$, $f_N(x)$ be the discretisation of f on Λ_N. Then

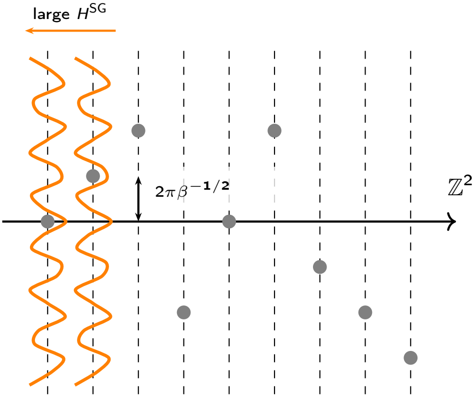
$$\lim_{N\to\infty} \log \mathbb{E}^{\mathrm{DG}}_{\Lambda_N,\beta}[e^{(f_N,\sigma)_{\mathbb{T}^2}}] = \frac{\beta(1+\mathsf{s}^{\mathrm{DG}}(\beta))^{-1}}{2}(f,(-\Delta_{\mathbb{T}^2})^{-1}f)_{\mathbb{T}^2}$$

with the same $\mathsf{s}^{\mathrm{DG}}(\beta) = O(e^{-c\beta})$ as in Theorem 1.

The lattice sine-Gordon model

Defn : $\Omega_N^{\mathrm{SG}} = \{\phi \in \mathbb{R}^{\Lambda_N} : \phi(0) = 0\}$, 'activity' $z\,(\ll 1)$

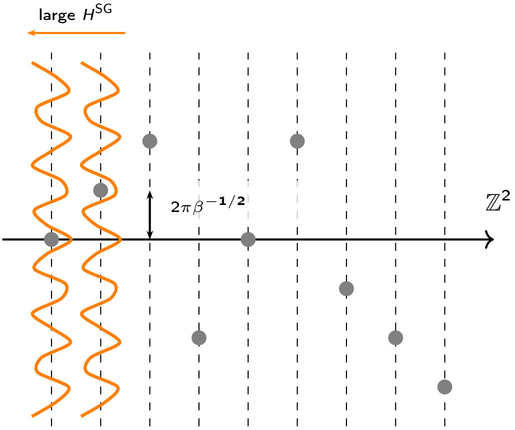
$$H_{N,\beta}^{\mathrm{SG}}(\phi) := \frac{1}{4}\sum_{x\sim y}(\phi(x)-\phi(y))^2 + z\underbrace{\sum_{x\in\Lambda_N}\cos(\beta^{1/2}\phi(x))}_{\mathrm{COS}(\Lambda_N,\beta^{1/2}\phi)}.$$



The lattice sine-Gordon model

A formal correspondence SG → DG

$$\mathbb{P}^{\mathsf{SG}}_{\Lambda_N,\beta} \xrightarrow{z\to-\infty} \mathbb{P}^{\mathsf{DG}}_{\Lambda_N,\beta}$$

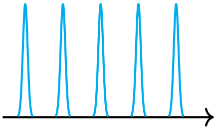


The lattice sine-Gordon model

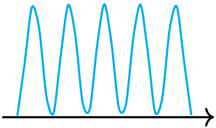
A formal correspondence DG $\to$ SG

$$
\begin{aligned}
Z^{\mathrm{DG}}_{N,\beta} &= \sum_{\sigma\in\Omega_{N,\beta}} e^{-H^{\mathrm{DG}}_N(\sigma)} \\
&= \int_{\Omega^{\mathrm{SG}}_N} d\phi \prod_{x\in\Lambda_N} \sum_{n\in 2\pi\mathbb{Z}/\sqrt{\beta}} \delta_n(\phi(x)) e^{-\frac{1}{2}(\phi,-\Delta\phi)}
\end{aligned}
$$

$$
\left(\sum_{n\in 2\pi\mathbb{Z}/\sqrt{\beta}} \delta_n(\phi) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{q\geq 1}\cos(q\beta^{1/2}\phi) = \frac{1}{2\pi} e^{z\cos(\beta^{1/2}\phi)} + (\text{higher } q) \right)
$$



$=$



$+$ (higher q terms)

The lattice sine-Gordon model

A formal correspondence DG $\to$ SG

$$
\begin{aligned}
Z^{\mathrm{DG}}_{N,\beta} &= \int_{\Omega_N^{\mathrm{SG}}} d\phi \prod_{x\in\Lambda_N} \sum_{n\in 2\pi\mathbb{Z}/\sqrt{\beta}} \delta_n(\phi(x)) e^{-\frac{1}{2}(\phi,-\Delta\phi)} \\
&\propto \mathbb{E}^{\phi}\Big[\prod_{x\in\Lambda_N} \sum_{n\in 2\pi\mathbb{Z}/\sqrt{\beta}} \delta_n(\phi(x))\Big], \qquad \phi \sim \mathcal{N}(0,(-\Delta)^{-1})
\end{aligned}
$$

- Covariance decomposition : $\exists \gamma > 0$ such that

$$
\begin{aligned}
& C(\gamma) := (-\Delta)^{-1} - \gamma\,\mathrm{id} \geq 0 \\
\Rightarrow \quad & \phi = \varphi + \sqrt{\gamma}\,Y, \;\; (\varphi, Y) \sim \mathcal{N}(0, C(\gamma) \oplus \mathrm{id}).
\end{aligned}
$$

Then

$$
\mathbb{E}^{\phi}[F(\phi)] = \mathbb{E}^{\varphi}\mathbb{E}^{Y}[F(\varphi + \sqrt{\gamma}\,Y)]
$$

The lattice sine-Gordon model

A formal correspondence DG $\to$ SG

$$Z^{\mathrm{DG}}_{N,\beta} \propto \mathbb{E}^{\phi}\Big[\prod_{x\in\Lambda_N}\sum_{n\in 2\pi\mathbb{Z}/\sqrt{\beta}}\delta_n(\phi(x))\Big], \qquad \phi \sim \mathcal{N}(0,(-\Delta)^{-1})$$

$$\big(\phi = \varphi + \sqrt{\gamma}Y, \ \ (\varphi, Y) \sim \mathcal{N}(0, C(\gamma)\oplus \mathrm{id})\big)$$

$$= \mathbb{E}^{\varphi}\mathbb{E}^{Y}\Big[\prod_{x\in\Lambda_N}\sum_{n\in 2\pi\mathbb{Z}/\sqrt{\beta}}\delta_n(\varphi(x)+\sqrt{\gamma}Y(x))\Big]$$

$\big(Y(x)$ are independent $\big)$

$$= \mathbb{E}^{\varphi}\prod_{x\in\Lambda_N}\mathbb{E}^{Y(x)}\Big[\sum_{n\in 2\pi\mathbb{Z}/\sqrt{\beta}}\delta_n(\varphi(x)+\sqrt{\gamma}Y(x))\Big] \leftarrow \text{1 dim'l integral}$$

$$(\varphi, Y(x)) \sim \mathcal{N}(0, C(\gamma)\oplus 1)$$

The lattice sine-Gordon model

A formal correspondence DG $\to$ SG

- 1 dimensional integral formula

$$Z^{\mathrm{DG}}_{N,\beta} \propto \mathbb{E}^{\varphi} \prod_{x \in \Lambda_N} \mathbb{E}^{Y}\Big[\sum_{n \in 2\pi\mathbb{Z}/\sqrt{\beta}} \delta_n(\varphi(x) + \sqrt{\gamma}Y) \Big]$$

$$(\varphi, Y) \sim \mathcal{N}(0, C(\gamma) \oplus 1).$$

The lattice sine-Gordon model

A formal correspondence DG $\to$ SG

- 1 dimensional integral formula

$$Z_{N,\beta}^{\mathrm{DG}} \propto \mathbb{E}^{\varphi} \prod_{x \in \Lambda_N} \mathbb{E}^{Y} \Big[\sum_{n \in 2\pi\mathbb{Z}/\sqrt{\beta}} \delta_n(\varphi(x) + \sqrt{\gamma} Y) \Big]$$

$$(\varphi, Y) \sim \mathcal{N}(0, C(\gamma) \oplus 1).$$

- **Lemma 2.2.** If $F(y) := \mathbb{E}^Y[\sum_{n \in 2\pi\mathbb{Z}/\sqrt{\beta}} \delta_n(y + \sqrt{\gamma} Y)]$, $Y \in \mathcal{N}(0,1)$, then

$$F(y) = c(\gamma, \beta)\Big(1 + \sum_{q=1}^{\infty} 2e^{-\frac{\gamma\beta}{2}q^2} \cos(q\beta^{1/2} y)\Big).$$

The lattice sine-Gordon model

Proposition (DG → SG)

$$Z^{\mathrm{DG}}_{N,\beta} \propto \mathbb{E}^{\varphi}\Big[\prod_{x\in\Lambda_N} F(\varphi(x))\Big], \quad \varphi \sim \mathcal{N}(0, (-\Delta)^{-1} - \gamma\,\mathrm{id})$$

where $F(y) = c(\gamma,\beta)(1 + \sum_{q=1}^{\infty} 2e^{-\frac{\gamma\beta}{2}q^2}\cos(q\beta^{1/2}y))$.

In other words, with $z = O(e^{-c\beta})$,

$$Z^{\mathrm{DG}}_{N,\beta} \propto \int_{\Omega_N^{\mathrm{SG}}} d\varphi\, \underbrace{e^{-\frac{1}{2}(\varphi,((-\Delta)^{-1}-\gamma\,\mathrm{id})^{-1}\varphi)}}_{\text{"gradient interaction"}}\, \underbrace{e^{z\,\mathrm{COS}(\Lambda_N,\varphi)+(\text{higher } q \text{ terms})}}_{\text{"sine-Gordon potential"}}.$$

The lattice sine-Gordon model

$$Z^{\mathrm{DG}}_{N,\beta} \propto \text{"} \int_{\Omega^{\mathrm{SG}}_N} d\phi e^{-H^{\mathrm{SG}}_{N,\beta}(\phi) + (\text{"higher } q \text{ terms"})} \text{"}$$

Remaining of the slides :

(1) Why don't "higher q's" matter? Dual picture, the Coulomb gas.

(2) Renormalisation group method on sine-Gordon.

(3) Why don't "higher q's" matter? Renormalisation group picture, coming back to the Discrete Gaussian model.

The lattice sine-Gordon model

$$Z^{\mathrm{DG}}_{N,\beta} \propto \text{"} \int_{\Omega_N^{\mathrm{SG}}} d\phi \, \underbrace{e^{-H^{\mathrm{SG}}_{N,\beta}(\phi) + (\text{"higher } q \text{ terms"})}}_{\exp\left(-\bar{H}^{\mathrm{SG}}\right)} \text{"}$$

(1) **Why don't "higher q's" matter? Dual picture, the Coulomb gas.**

(2) Renormalisation group method on sine-Gordon.

(3) Why don't "higher q's" matter? Renormalisation group picture, coming back to the Discrete Gaussian model.

The lattice sine-Gordon model

Why don't "higher q's" matter?

(1) Dual picture, the Coulomb gas.

- A generalised sine-Gordon model has

$$\bar{H}^{\mathrm{SG}}_{N,\beta}(\phi) = \frac{1}{4}\sum_{x\sim y}(\phi_x - \phi_y)^2 - \sum_{x\in\Lambda_N}\log\Big(1 + 2\sum_{q\geq 1} z_q \cos(q\beta^{1/2}\phi(x))\Big).$$

Then with $z_q = z_{-q}$,

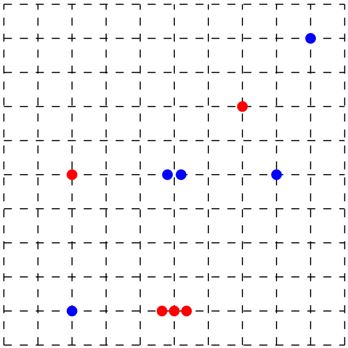
$$\begin{aligned}\bar{Z}_{N,\beta} &= \int_{\Omega_N^{\mathrm{SG}}} d\phi e^{-\frac{1}{2}(\phi,-\Delta\phi)} \prod_{x\in\Lambda_N}\sum_{q\in\mathbb{Z}} z_q e^{iq\beta^{1/2}\phi(x)} \\ &= \sum_{\boldsymbol{q}\in\mathbb{Z}^{\Lambda_N}}\prod_x z_{\boldsymbol{q}(x)}\int_{\Omega_N^{\mathrm{SG}}} d\phi e^{-\frac{1}{2}(\phi,-\Delta\phi)+i\beta^{1/2}(\phi,\boldsymbol{q})} \\ &= \sum_{\boldsymbol{q}} 1_{\sum_x \boldsymbol{q}(x)=0} z_{\boldsymbol{q}} \underbrace{e^{-\frac{\beta}{2}(\boldsymbol{q},(-\Delta)^{-1}\boldsymbol{q})}}_{\text{Coulomb interaction}} \quad (\leftarrow \ q : \text{charge configuration})\end{aligned}$$

The lattice sine-Gordon model

Why don't "higher q's" matter?

(1) Dual picture, the Coulomb gas.

$$\bar{Z}_{N,\beta} = \sum_{\boldsymbol{q}} 1_{\sum_x \boldsymbol{q}(x)=0} \overbrace{z_{\boldsymbol{q}}}^{\text{'activity'}} \underbrace{e^{-\frac{\beta}{2}(\boldsymbol{q},(-\Delta)^{-1}\boldsymbol{q})}}_{\text{Coulomb interaction}}$$



● $q_x = +1$ ● $q_x = -1$

The lattice sine-Gordon model

$$Z^{\mathrm{DG}}_{N,\beta} \propto \text{"} \int_{\Omega^{\mathrm{SG}}_N} d\phi \, \underbrace{e^{-H^{\mathrm{SG}}_{N,\beta}(\phi) + (\text{"higher } q \text{ terms"})}}_{\exp\left(-\bar{H}^{\mathrm{SG}}\right)} \text{"}$$

(1) Why don't "higher q's" matter? Dual picture, the Coulomb gas.

(2) **Renormalisation group method on sine-Gordon.**

(3) Why don't "higher q's" matter? Renormalisation group picture, coming back to the Discrete Gaussian model.

Scaling limit of sine-Gordon

$$\mathbb{P}^{\mathsf{SG}}_{\Lambda_N,\beta}(\phi \in A) \propto \int_A d\phi e^{-H^{\mathsf{SG}}_{\Lambda_N,\beta}(\phi)},$$

$$H^{\mathsf{SG}}_{\Lambda_N,\beta}(\phi) = \frac{1}{2}(\phi, -\Delta\phi) + z\,\mathsf{COS}(\Lambda_N, \beta^{1/2}\phi),$$

Scaling limit of sine-Gordon

$$\mathbb{P}^{\mathsf{SG}}_{\Lambda_N,\beta}(\phi \in A) \propto \int_A d\phi e^{-H^{\mathsf{SG}}_{\Lambda_N,\beta}(\phi)},$$

$$H^{\mathsf{SG}}_{\Lambda_N,\beta}(\phi) = \frac{1}{2}(\phi, -\Delta\phi) + z\,\mathsf{COS}(\Lambda_N, \beta^{1/2}\phi),$$

Theorem 4

"Scaling limit of (non-general) sine-Gordon is GFF on $\mathbb{T}^2$"

Let $\beta > 0$ sufficiently large and $|z|$ small, $f \in C^\infty(\mathbb{T}^2)$ be such that $\int_{\mathbb{T}^2} f = 0$, and for each $N > 0$, $f_N(x)$ be the discretisation of f on Λ_N. Then

$$\lim_{N\to\infty} \log \mathbb{E}^{\mathsf{SG}}_{\Lambda_N,\beta}[e^{(f_N,\phi)_{\mathbb{T}^2}}] = \underbrace{\frac{\beta(1+\mathbf{s}^{\mathsf{SG}}(\beta,z))^{-1}}{2}(f,(-\Delta_{\mathbb{T}^2})^{-1}f)_{\mathbb{T}^2}}_{\text{GFF with covariance } \beta(1+\mathbf{s}^{\mathsf{SG}})(-\Delta)^{-1}}$$

for some $\mathbf{s}^{\mathsf{SG}}(\beta,z) = O(|z|)$.

Theorem 4

Let $\beta > 0$ sufficiently large and $|z|$ small, $f \in C^\infty(\mathbb{T}^2)$ be such that $\int_{\mathbb{T}^2} f = 0$, and for each $N > 0$, $f_N(x)$ be the discretisation of f on Λ_N. Then

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Theorem 4

Let $\beta > 0$ sufficiently large and $|z|$ small, $f \in C^\infty(\mathbb{T}^2)$ be such that $\int_{\mathbb{T}^2} f = 0$, and for each $N > 0$, $f_N(x)$ be the discretisation of f on Λ_N. Then

$$\lim_{N\to\infty} \log \mathbb{E}^{\mathrm{SG}}_{\Lambda_N,\beta}[e^{(f_N,\phi)_{\mathbb{T}^2}}] = \underbrace{\frac{\beta(1+\mathbf{s}^{\mathrm{SG}}(\beta,z))^{-1}}{2}(f,(-\Delta_{\mathbb{T}^2})^{-1}f)_{\mathbb{T}^2}}_{\text{GFF with covariance } \beta(1+\mathbf{s}^{\mathrm{SG}})(-\Delta)^{-1}}$$

for some $\mathbf{s}^{\mathrm{SG}}(\beta,z) = O(|z|)$.

- Expecting this :

$$\langle e^{(f,\phi)}\rangle^{\mathrm{SG}}_{\Lambda,\beta} \propto \int_{\Omega_N^{\mathrm{SG}}} d\phi e^{-\frac{1}{2}(\phi,-\Delta\phi)+\sqrt{\beta}(f,\phi)+z\sum_{x\in\Lambda_N}\cos(\beta^{1/2}\phi(x))}$$

$$= \int_{\Omega_N^{\mathrm{SG}}} d\phi \underbrace{e^{-\frac{1+\mathbf{s}^{\mathrm{SG}}}{2}(\phi,-\Delta\phi)+\sqrt{\beta}(f,\phi)}}_{\text{mgf of Gaussian}} \left(e^{\frac{\mathbf{s}^{\mathrm{SG}}}{2}(\phi,-\Delta\phi)+z\sum_{x\in\Lambda_N}\cos(\beta^{1/2}\phi(x))}\right)$$

RG on sine-Gordon

Preliminary steps 1

- Stiffness renormalisation

$$\langle e^{(f,\phi)}\rangle \propto \int_{\Omega_N^{\mathrm{SG}}} d\phi e^{-\frac{1+\mathbf{s}^{\mathrm{SG}}}{2}(\phi,-\Delta\phi)+\sqrt{\beta}(f,\phi)} e^{\frac{\mathbf{s}^{\mathrm{SG}}}{2}(\phi,-\Delta\phi)+z\sum_{x\in\Lambda_N}\cos(\beta^{1/2}\phi(x))}$$

$$\left(\text{rescaling } \varphi = \sqrt{1+\mathbf{s}^{\mathrm{SG}}}\phi,\ \beta' = \beta/(1+\mathbf{s}^{\mathrm{SG}}),\ s_0 = \mathbf{s}^{\mathrm{SG}}/(1+\mathbf{s}^{\mathrm{SG}})\right)$$

$$\propto \int_{\Omega_N^{\mathrm{SG}}} d\varphi e^{-\frac{1}{2}(\varphi,-\Delta\varphi)+\sqrt{\beta'}(f,\varphi)} \underbrace{e^{\frac{s_0}{2}(\varphi,-\Delta\varphi)+z\,\mathrm{COS}_0(\Lambda_N,(\beta')^{1/2}\varphi)}}_{Z_0(\varphi)}$$

(denoted $\mathrm{COS}_j(\Lambda_N,(\beta')^{1/2}\varphi) = L^{-2j}\sum_{x\in\Lambda_N}\cos((\beta')^{1/2}\varphi(x))$).

RG on sine-Gordon

Preliminary steps 2

- Moment generating function : let $\sum_x f(x) = 0$ (so $f \in \mathrm{Im}(-\Delta))$

$$\langle e^{(f,\phi)} \rangle^{\mathrm{SG}}_{\Lambda,\beta} \propto \int_{\Omega_N^{\mathrm{SG}}} d\varphi e^{-\frac{1}{2}(\varphi,-\Delta\varphi)+\sqrt{\beta'}(f,\varphi)} Z_0(\varphi)$$

(Completion of square, $g = \beta^{-1/2}(-\Delta)^{-1} f$)

$$= e^{\frac{\beta'}{2}(f,(-\Delta)^{-1}f)} \int_{\Omega_N^{\mathrm{SG}}} d\varphi e^{-\frac{1}{2}\left(\varphi-g,-\Delta(\varphi-g)\right)} Z_0(\varphi)$$

$$\propto e^{\frac{\beta'}{2}(f,(-\Delta)^{-1}f)} \mathbb{E}^{\varphi}[Z_0(\varphi+g)]$$

where $\varphi \sim \mathcal{N}(0,(-\Delta)^{-1})$, $\beta' = \beta/(1+\mathbf{s}^{\mathrm{SG}})$.

Theorem 4

Let $\beta > 0$ sufficiently large, $f \in C^\infty(\mathbb{T}^2)$ be such that $\int_{\mathbb{T}^2} f = 0$, and for each $N > 0$, $f_N(x)$ be the discretisation of f on Λ_N. Then

$$\lim_{N \to 0} \log \mathbb{E}_{\Lambda_N,\beta}[e^{(f_N,\phi)_{\mathbb{T}^2}}] = \underbrace{\frac{\beta(1+\mathbf{s}^{\mathrm{SG}}(\beta))^{-1}}{2}(f,(-\Delta_{\mathbb{T}^2})^{-1}f)_{\mathbb{T}^2}}_{\text{GFF with covariance } \beta(1+\mathbf{s}^{\mathrm{SG}})(-\Delta)^{-1}}$$

for some $\mathbf{s}^{\mathrm{SG}} = O(|z|)$.

- Have : $\langle e^{(f,\phi)} \rangle_{\Lambda_N,\beta} \propto e^{\frac{\beta'}{2}(f,(-\Delta)^{-1}f)} \mathbb{E}^{\varphi}[Z_0(\varphi+g)]$.

Goal

$$\frac{\mathbb{E}^{\varphi}[Z_0(\varphi+g)]}{\mathbb{E}^{\varphi}[Z_0(\varphi)]} \to 1 \text{ as } N \to \infty \text{ (i.e. } \Lambda_N \to \mathbb{Z}^2).$$

RG on sine-Gordon

Goal

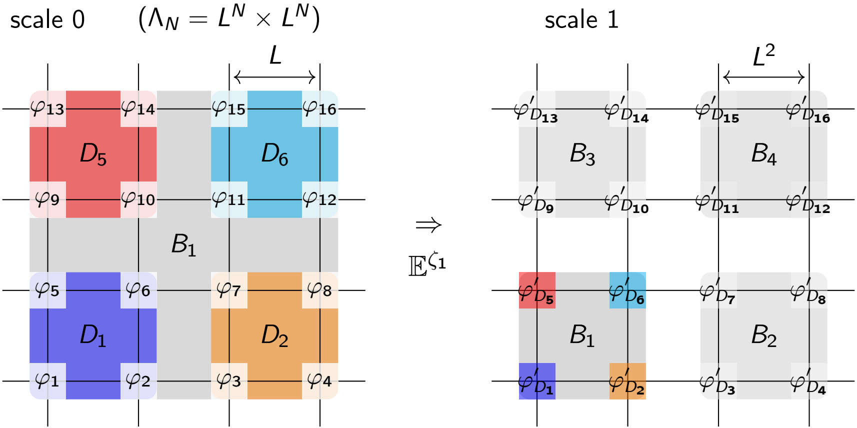
Controlling $\mathbb{E}^{\varphi}[Z_0(\varphi+g)]$ as $N \to \infty$

$$Z_0(\varphi) = 1 + \frac{s_0}{2}|\nabla\varphi|^2_{\Lambda_N} + z\,\mathrm{COS}_0(\Lambda_N, (\beta')^{1/2}\varphi) + |\Lambda_N| o(s_0, z)$$

$$\Downarrow$$

$$\begin{aligned}\mathbb{E}^{\varphi}[Z_0(\varphi+g)] = 1 + \frac{s_0}{2}|\Lambda_N| + \frac{s_0}{2}|\nabla g|^2_{\Lambda_N} \\ + z e^{-\frac{\beta'}{2}(-\Delta)_{00}} L^{2N}\,\mathrm{COS}_N(\Lambda_N, (\beta')^{1/2} g) \\ + |\Lambda_N| \underbrace{o(s_0, z)}_{\text{not small enough}} .\end{aligned}$$

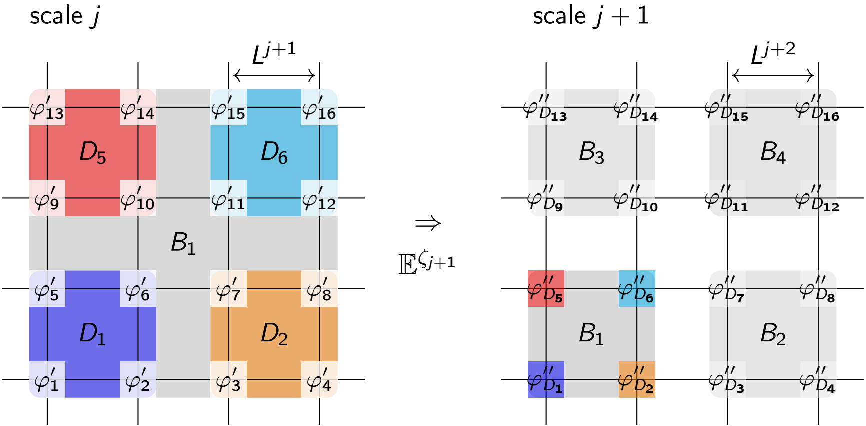
RG on sine-Gordon



(1) Field decomposition : $\varphi = \varphi' + \zeta_1$, ζ_1 consisting of fluctuations of φ in length scale $< L$.

(2) Reblocking.

RG on sine-Gordon



(1) Field decomposition : $\varphi' = \varphi'' + \zeta_{j+1}$, ζ_{j+1} consisting of fluctuations of φ' in length scale $< L^{j+1}$.

(2) Reblocking.

RG on sine-Gordon

Field / Covariance decomposition of GFF

- Want : $\varphi = \sum_{j=1}^{N} \zeta_j$ where

$$(-\Delta)^{-1} = \sum_{j=1}^{N} \Gamma_j, \quad (\zeta_j)_{1\le j\le N} \sim \mathcal{N}(0, \bigoplus_{1\le j\le N} \Gamma_j)$$

$$\text{s.t. } \Gamma_j(x,y) = 0 \quad \forall |x-y|_\infty > L^j.$$

  called a **finite range decomposition**.

- A good guess :

$$(-\Delta)^{-1}(x,y) = \frac{1}{|\Lambda_N|} \sum_{k\in\Lambda_N^*} e^{ik\cdot(y-x)} (-\hat{\Delta})^{-1}(k)$$

$$\Rightarrow \quad \Gamma_j(x,y) \simeq \frac{1}{|\Lambda_N|} \sum_{k\in\Lambda_N^*} e^{ik\cdot(y-x)} (-\hat{\Delta})^{-1}(k) 1_{|k|\in[L^{-j}, L^{-j+1}]}$$

RG on sine-Gordon

Field / Covariance decomposition of GFF

- A good guess : with $-\hat{\Delta}(k) \simeq |k|^2$,

$$\Gamma_j(x,y) \simeq \frac{1}{|\Lambda_N|} \sum_{k \in \Lambda_N^*} e^{ik\cdot(y-x)} (-\hat{\Delta})^{-1}(k) 1_{|k| \in [L^{-j}, L^{-j+1}]}$$

1. When $|x-y|_\infty \leq L^{j-1}$:

$$|\Gamma_j(0,x) - \Gamma_j(0,y)| \lesssim C|x-y|_\infty |\nabla \Gamma_j|_\infty \lesssim L^{-1}.$$

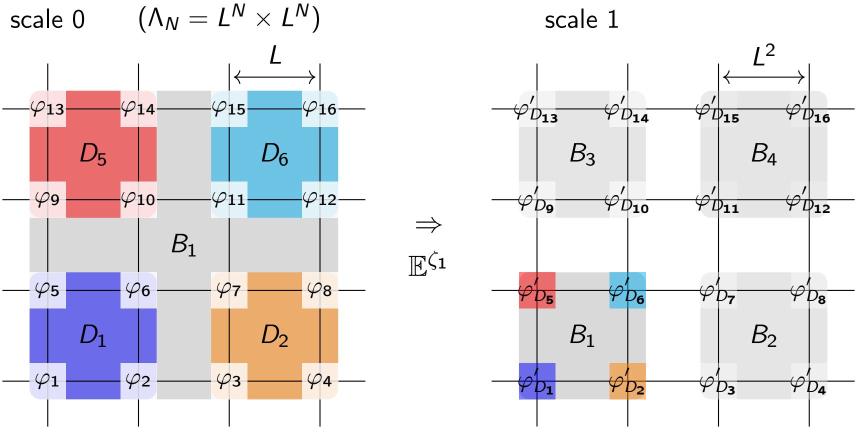
2. For $|x|_\infty > L^j$:

$$|\Gamma_j(0,x)| \simeq 0$$

3. In limit $N \to \infty$:

$$\Gamma_j(0,0) \simeq \frac{1}{2\pi} \log L \quad \Rightarrow \quad \textbf{(fluctuations are distributed well)}$$

RG on sine-Gordon



(1) Field decomposition : $\varphi = \varphi' + \zeta_1$, ζ_1 consisting of fluctuations of φ in length scale $< L$.

(2) Reblocking.

RG on sine-Gordon

Action of $\mathbb{E}^{\zeta_1}$ on $Z_0(\varphi_1 + \zeta_1)$

$$Z_0(\varphi_1 + \zeta_1) = 1 + \frac{s_0}{2}|\nabla\varphi|^2 + z\,\mathrm{COS}_0(\Lambda_N, (\beta')^{1/2}\varphi) + |\Lambda_N| o(s_0, z).$$

- On gradient term :

$$\mathbb{E}^{\zeta_1}|\nabla(\varphi_1 + \zeta_1)|^2_{\Lambda_N} = \delta E_1 |\Lambda_N| + |\nabla\varphi_1|^2_{\Lambda_N}$$

- On cosine term :

$$\begin{aligned}
&\mathbb{E}^{\zeta_1}\,\mathrm{COS}_0(\Lambda_N, (\beta')^{1/2}(\varphi_1 + \zeta_1)) \\
&\quad = L^2 e^{-\frac{\beta'}{2}\Gamma_1(0,0)}\,\mathrm{COS}_1(\Lambda_N, (\beta')^{1/2}\varphi_1) \\
&\quad \simeq L^{2-\frac{\beta'}{4\pi}}\,\mathrm{COS}_1(\Lambda_N, (\beta')^{1/2}\varphi_1) \quad \left(\Gamma_1(0,0) \simeq \frac{1}{2\pi}\log L\right)
\end{aligned}$$

RG on sine-Gordon

Summary

- Field / Covariance decomposition : $\varphi = \sum_{j=1}^{N} \zeta_j$ where

$$(-\Delta)^{-1} = \sum_{j=1}^{N} \Gamma_j, \quad (\zeta_j)_{1 \le j \le N} \sim \mathcal{N}(0, \bigoplus_{1 \le j \le N} \Gamma_j)$$

- Upon each $\mathbb{E}^{\zeta_{j+1}}$,

$$\begin{aligned}\log Z_1(\varphi_1) &:= \log \mathbb{E}^{\zeta_1}[Z_0(\varphi_1 + \zeta_1)] \\ &= E_1|\Lambda_N| + \frac{s_1}{2}|\nabla\varphi_1|^2 + z_1 \,\mathrm{COS}_1(\Lambda_N, (\beta')^{1/2}\varphi_1) + (\text{error})\end{aligned}$$

$$\begin{aligned}\log Z_{j+1}(\varphi_{j+1}) &:= \log \mathbb{E}^{\zeta_{j+1}}[Z_j(\varphi_{j+1} + \zeta_{j+1})] \\ &= E_{j+1}|\Lambda_N| + \frac{s_{j+1}}{2}|\nabla\varphi_{j+1}|^2 + z_{j+1} \,\mathrm{COS}_{j+1}(\Lambda_N, (\beta')^{1/2}\varphi_{j+1}) \\ &\quad + (\text{error})\end{aligned}$$

RG on sine-Gordon

On the error

$$Z_j(\varphi) \propto e^{\frac{s_j}{2}|\nabla\varphi|^2 + z_j \cos_j(\Lambda_N,(\beta')^{1/2}\varphi)}\big(1 + (\text{error})\big)$$

- Ideally,

$$1 + (\text{error}) = \text{"} \prod_{B \in \text{scale j block}} \big(1 + K_j(B,\varphi)\big)\text{"}.$$

But this form is not stable under $\mathbb{E}^{\zeta_{j+1}}$.

RG on sine-Gordon

On the error

$$Z_j(\varphi) \propto e^{\frac{s_j}{2}|\nabla\varphi|^2 + z_j \cos_j(\Lambda_N, (\beta')^{1/2}\varphi)} \big(1 + (\text{error})\big)$$

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  But this form is not stable under $\mathbb{E}^{\zeta_{j+1}}$.
- Error is **parametrised by K_j,**

$$(\text{error}) = \sum_{X \subset \text{scale j block}} e^{-\frac{s_j}{2}(\nabla\varphi, \nabla\varphi)_X^2 - z_j \cos_j(X, (\beta')^{1/2}\varphi)} \overbrace{K_j(X, \varphi)}^{\text{polymer activities}}$$

RG on sine-Gordon

Theorem

If $\log Z_j(\varphi) = \frac{s_j}{2}|\nabla\varphi|^2 + z_j \operatorname{COS}_j(\Lambda_N, (\beta')^{1/2}\varphi) + F(K_j)$, then

$$\begin{aligned} &s_{j+1} = s_j + e_j(K_j), \quad z_{j+1} \simeq L^2 L^{-\frac{\beta'}{4\pi}} z_j, \\ &K_{j+1} = L^{-\alpha} O(K_j) + o(s_j, z_j, K_j) \end{aligned}$$

with initial conditions $s_0, z_0 = z$.

- If $\beta' > 8\pi$, $\exists\ s_0$ such that

$$(s_j, z_j, K_j) \xrightarrow[j\to\infty]{} (0,0,0) \quad \Leftrightarrow \quad \mathbb{E}^{\varphi}[Z_0(\varphi + g)] \sim C_N^{|\Lambda_N|}$$

  Since $s_0 = \frac{\mathbf{s}^{\mathrm{SG}}}{1+\mathbf{s}^{\mathrm{SG}}}$,

$$\mathbf{s}^{\mathrm{SG}} = \frac{s_0}{1-s_0}.$$

RG on sine-Gordon

Theorem 4 (scaling limit of sine-Gordon)

Let $\beta > 0$ sufficiently large and $|z|$ small, $f \in C^\infty(\mathbb{T}^2)$ be such that $\int_{\mathbb{T}^2} f = 0$, and for each $N > 0$, $f_N(x)$ be the discretisation of f on Λ_N. Then

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for some $\mathsf{s}^{\mathsf{SG}} = O(|z|)$.

RG on sine-Gordon

Theorem

If $\log Z_j(\varphi) = \frac{s_j}{2}|\nabla\varphi|^2 + z_j \, \mathrm{COS}_j(\Lambda_N, (\beta')^{1/2}\varphi) + F(K_j)$, then

$$s_{j+1} = s_j + e_j(K_j), \quad z_{j+1} \simeq L^2 L^{-\frac{\beta'}{4\pi}} z_j.$$

RG on sine-Gordon

Theorem
If $\log Z_j(\varphi) = \frac{s_j}{2}|\nabla\varphi|^2 + z_j \operatorname{COS}_j(\Lambda_N, (\beta')^{1/2}\varphi) + F(K_j)$, then

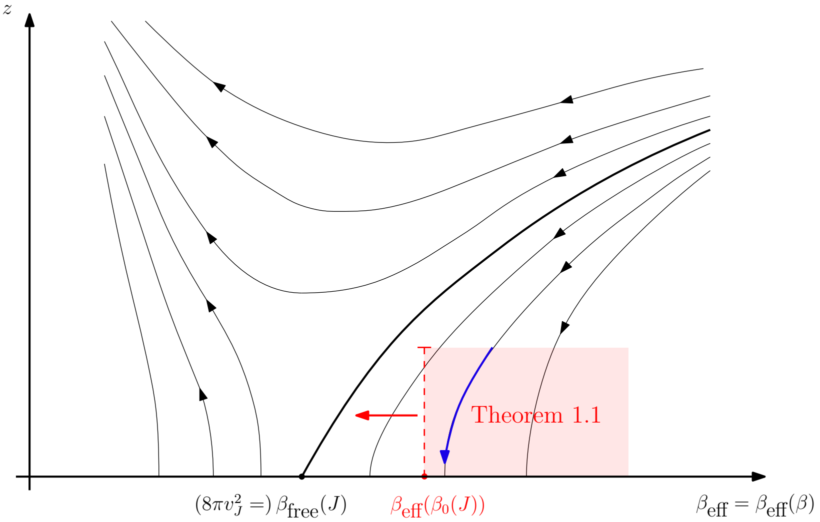
$$s_{j+1} = s_j + e_j(K_j), \quad z_{j+1} \simeq L^2 L^{-\frac{\beta'}{4\pi}} z_j.$$

Theorem [Falco, 2012]
Using second order analysis,

$$s_{j+1} = s_j - a_j z_j^2 + e_j^1, \quad z_{j+1} \simeq L^2 L^{-\frac{\beta'}{4\pi}} (z_j - b_j s_j z_j + e_j^2),$$

i.e., satisfies the **Kosterlitz-Thouless flow equation**.

RG on sine-Gordon



Connecting sine-Gordon $\to$ dgauss

Theorem 2 [I, B.-P.-R., 2022]

"Scaling limit of dgauss is a Gaussian free field on $\mathbb{T}^2$"

Let $\beta > 0$ sufficiently large, $f \in C^\infty(\mathbb{T}^2)$ be such that $\int_{\mathbb{T}^2} f = 0$, and for each $N > 0$, $f_N(x)$ be the discretisation of f on Λ_N. Then

$$\lim_{N\to\infty} \log \mathbb{E}^{\mathrm{DG}}_{\Lambda_N,\beta}[e^{(f_N,\sigma)_{\mathbb{T}^2}}] = \frac{\beta(1+\mathsf{s}^{\mathrm{DG}}(\beta))^{-1}}{2}(f,(-\Delta_{\mathbb{T}^2})^{-1}f)_{\mathbb{T}^2}$$

with the same $\mathsf{s}^{\mathrm{DG}}(\beta) = O(e^{-c\beta})$ as in Theorem 1.

The lattice sine-Gordon model

$$Z_{N,\beta}^{\mathrm{DG}} \propto \text{"} \int_{\Omega_N^{\mathrm{SG}}} d\phi e^{-H_{N,\beta}^{\mathrm{SG}}(\phi) + (\text{"higher } q \text{ terms"})} \text{"}$$

(1) Why don't "higher q's" matter? Dual picture, the Coulomb gas.

(2) Renormalisation group method on sine-Gordon.

(3) **Why don't "higher q's" matter? Renormalisation group picture, coming back to the Discrete Gaussian model.**

Connecting sine-Gordon $\to$ dgauss

Why don't "higher q's" matter?

(2) Renormalisation group perspective.

- Recall on cosine term, $\beta' > 8\pi$:

$$\mathbb{E}^{\zeta_1}\, \mathrm{COS}_0(\Lambda_N, (\beta')^{1/2}(\varphi_1 + \zeta_1))$$

$$\simeq L^{2-\frac{\beta'}{4\pi}}\, \mathrm{COS}_1(\Lambda_N, (\beta')^{1/2}\varphi_1) \quad (\Gamma_1(0,0) \simeq \frac{1}{2\pi}\log L)$$

$$\Rightarrow \quad z_{j+1} \simeq L^{2-\frac{\beta'}{4\pi}} z_j$$

Connecting sine-Gordon $\to$ dgauss

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$$\Rightarrow \quad z_{j+1} \simeq L^{2-\frac{\beta'}{4\pi}} z_j$$

- Similarly for $q \geq 2$:

$$\mathbb{E}^{\zeta_1} \mathrm{COS}_0(\Lambda_N, q(\beta')^{1/2}(\varphi_1 + \zeta_1))$$

$$\simeq L^{2-\frac{\beta'}{4\pi}q^2} \mathrm{COS}_1(\Lambda_N, q(\beta')^{1/2}\varphi_1) \quad \Leftarrow \quad \text{"irrelevant"}$$

and decays exponentially faster than $\mathrm{COS}(\Lambda_N, (\beta')^{1/2}\varphi)$.

Connecting sine-Gordon $\to$ dgauss

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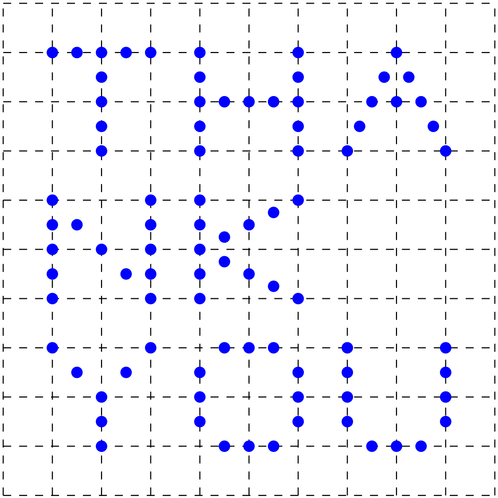

Figure 1: Lattice Thank You