2D Discrete Gaussian model at high temperature

- The infinite-volume scaling limit -

Jiwoon Park

2nd March, 2022

Based on the works with Roland Bauerschmidt and Pierre-François Rodriguez

- The Discrete Gaussian model, I. Renormalisation group flow at high temperature, arXiv:2202.02286 (2022)
- The Discrete Gaussian model, II. Infinite-volume scaling limit at high temperature, arXiv:2202.02287 (2022)

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- **Hamiltonian** $H_N^{\mathrm{DG}}(\sigma) = \frac{1}{4}\sum_{x\sim y\in\Lambda_N}(\sigma_x - \sigma_y)^2$ and

$$\mathbb{P}^{\mathrm{DG}}_{\Lambda_N,\beta}(\{\sigma\}) = \exp\Big(-\frac{1}{\beta}H_N^{\mathrm{DG}}(\sigma)\Big) \;/\; Z^{\mathrm{DG}}_{\Lambda_N,\beta}$$

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- Also called $\mathbb{Z}$-Ferromagnet or Integer-Valued GFF.

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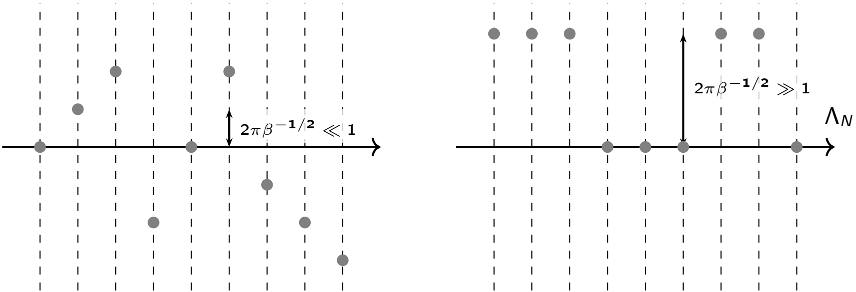
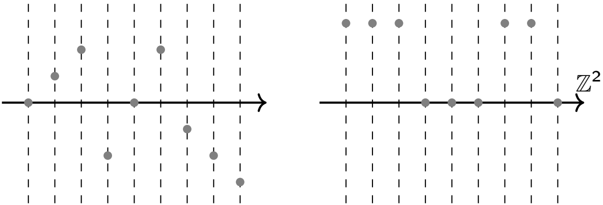


Two phases of dgauss model



"Delocalisation phase" "Localisation phase"

Kosterlitz-Thouless phase transition

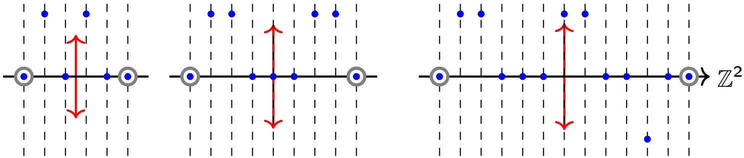
- (2D) XY model, Villain model, Coulomb gas, Sine-Gordon model

Two phases of dgauss model

Comparison of the Gibbs states

Investigation of the case of the 0-Dirichlet boundary condition :

- Localisation phase



bounded variance + strong clustering [BW'82]

$\Rightarrow$ translation invariant Gibbs measure with slope 0

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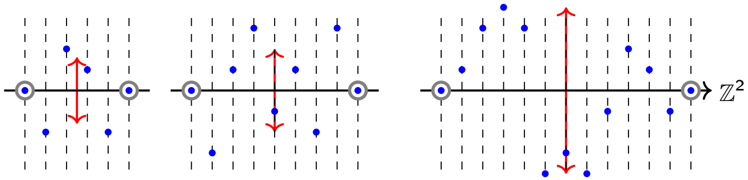

unbounded variance

+ non-existence of transl. inv. Gibbs measure with slope 0 [S'06]

$\Rightarrow$ has to consider gradient Gibbs states

Two phases of dgauss model

Delocalisation phase, $\beta \gg 1$

$f : \Lambda_N \to \mathbb{R}$ such that $\sum_{x\in\Lambda_N} f(x) = 0$,

- [Fröhlich-Park(1977)]

$$\mathbb{E}^{\mathrm{DG}}_{\Lambda_N,\beta}\left[e^{(f,\sigma)}\right] \le \mathbb{E}^{\mathrm{GFF}}_{\Lambda_N,\beta^{-1}}\left[e^{(f,\sigma)}\right] = e^{\frac{\beta}{2}(f,(-\Delta)^{-1}f)}$$

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- [Fröhlich-Spencer(1981)] There is $\epsilon_1(\beta) = o(1)$ (as $\beta \to \infty$) such that,

$$e^{\frac{\beta(1+\epsilon_1(\beta))}{2}(f,(-\Delta)^{-1}f)} = \mathbb{E}^{\mathrm{GFF}}_{\Lambda_N,\beta^{-1}(1+\epsilon_1(\beta))^{-1}}\left[e^{(f,\sigma)}\right] \leq \mathbb{E}^{\mathrm{DG}}_{\Lambda_N,\beta}\left[e^{(f,\sigma)}\right].$$

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- [Garban-Sepúlveda(2021)] With 0-Dirichlet boundary, if f has bounded support,

$$\mathrm{Var}^{\mathrm{DG}}_{\Lambda_N,\beta}[(f,\sigma)] \le \left(1 - 2\beta e^{-\frac{\beta}{2}+o(\beta)}\right)\mathrm{Var}^{\mathrm{GFF}}_{\Lambda_N,\beta^{-1}}[(f,\sigma)].$$

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- [Lammers-Ott(2021)] Discussion on dichotomy of the phases.

Delocalisation phase of dgauss model

Theorem 1 [II, B.-P.-R., 2022]

"Scaling limit of dgauss is a Gaussian free field on $\mathbb{R}^2$"

Let $\beta > 0$ **fixed** large, $f \in C_c^\infty(\mathbb{R}^2)$ be such that $\int_{\mathbb{R}^2} f = 0$, and for each $\epsilon > 0$, let $f_\epsilon(x)$ be the discretisation of $\epsilon^{-1} f(\epsilon x)$ on $\mathbb{Z}^2$. Then

$$\lim_{\epsilon \to 0} \log \mathbb{E}^{\mathrm{DG}}_{\mathbb{Z}^2,\beta}[e^{(f_\epsilon,\sigma)_{\mathbb{Z}^2}}] = \frac{\beta(1 + \mathsf{s}^{\mathrm{DG}}(\beta))^{-1}}{2} (f, (-\Delta_{\mathbb{R}^2})^{-1} f)_{\mathbb{R}^2}$$

where $\mathsf{s}^{\mathrm{DG}}(\beta) = O(e^{-c\beta})$, $c > 0$.

Delocalisation phase of dgauss model

Theorem 2 [I, B.-P.-R., 2022]

"Scaling limit of dgauss is a Gaussian free field on $\mathbb{T}^2$"

For $\beta > 0$ sufficiently large, $f \in C^\infty(\mathbb{T}^2)$ be such that $\int_{\mathbb{T}^2} f = 0$, and for each $N > 0$, $f_N(x)$ be the discretisation of f on Λ_N. Then

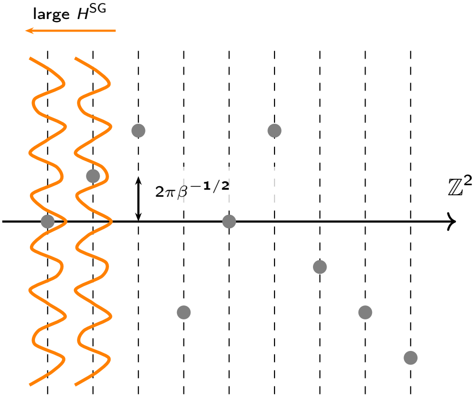
$$\lim_{N\to\infty} \log \mathbb{E}^{\mathrm{DG}}_{\Lambda_N,\beta}[e^{(f_N,\sigma)_{\mathbb{T}^2}}] = \frac{\beta(1+\mathsf{s}^{\mathrm{DG}}(\beta))^{-1}}{2}(f,(-\Delta_{\mathbb{T}^2})^{-1}f)_{\mathbb{T}^2}$$

with the same $\mathsf{s}^{\mathrm{DG}}(\beta) = O(e^{-c\beta})$ as in Theorem 1.

The lattice sine-Gordon model

Defn : $\Omega_N^{\mathrm{SG}} = \{\phi \in \mathbb{R}^{\Lambda_N} : \phi(0) = 0\}$, 'activity' $z\,(\ll 1)$

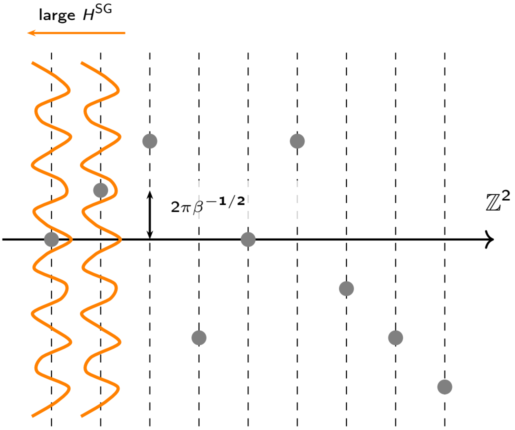
$$H_{N,\beta}^{\mathrm{SG}}(\phi) := \frac{1}{4}\sum_{x\sim y}(\phi(x)-\phi(y))^2 + z\underbrace{\sum_{x\in\Lambda_N}\cos(\beta^{1/2}\phi(x))}_{\mathrm{COS}(\Lambda_N,\beta^{1/2}\phi)}.$$



The lattice sine-Gordon model

A formal correspondence SG $\to$ DG

$$\mathbb{P}^{\mathsf{SG}}_{\Lambda_N,\beta} \xrightarrow{z\to-\infty} \mathbb{P}^{\mathsf{DG}}_{\Lambda_N,\beta}$$



The lattice sine-Gordon model

A formal correspondence DG $\to$ SG

$$
\begin{aligned}
Z^{\mathrm{DG}}_{N,\beta} &= \sum_{\sigma\in\Omega_{N,\beta}} e^{-H^{\mathrm{DG}}_N(\sigma)} \\
&= \int_{\Omega^{\mathrm{SG}}_N} d\phi \prod_{x\in\Lambda_N} \sum_{n\in 2\pi\mathbb{Z}/\sqrt{\beta}} \delta_n(\phi(x)) e^{-\frac{1}{2}(\phi,-\Delta\phi)}
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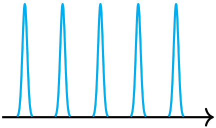
$$
\left(\sum_{n \in 2\pi\mathbb{Z}/\sqrt{\beta}} \delta_n(\phi) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{q \geq 1} \cos(q\beta^{1/2}\phi) = \frac{1}{2\pi} e^{z\cos(\beta^{1/2}\phi)} + (\text{higher } q) \right)
$$



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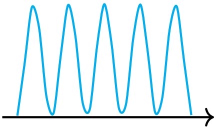

$+$ (higher q terms)

Scaling limit of sine-Gordon

$$Z^{\mathrm{DG}}_{N,\beta} \propto \text{"} \int_{\Omega^{\mathrm{SG}}_N} d\phi \, \underbrace{e^{-H^{\mathrm{SG}}_{N,\beta}(\phi) + (\text{"higher } q \text{ terms"})}}_{\exp\left(-\bar{H}^{\mathrm{SG}}\right)} \text{"}$$

Scaling limit of sine-Gordon

$$Z^{\mathrm{DG}}_{N,\beta} \propto \text{"} \int_{\Omega_N^{\mathrm{SG}}} d\phi \, \underbrace{e^{-H^{\mathrm{SG}}_{N,\beta}(\phi)+(\text{"higher } q \text{ terms"})}}_{\exp\left(-\bar{H}^{\mathrm{SG}}\right)} \text{"}$$

(1) **Renormalisation group method on sine-Gordon.**

(2) Why don't "higher q's" matter? Renormalisation group picture, coming back to the Discrete Gaussian model.

Scaling limit of sine-Gordon

$$\mathbb{P}^{\mathsf{SG}}_{\Lambda_N,\beta}(\phi \in A) \propto \int_A d\phi e^{-H^{\mathsf{SG}}_{\Lambda_N,\beta}(\phi)},$$

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Theorem 4

"Scaling limit of (non-general) sine-Gordon is GFF on $\mathbb{T}^2$"

Let $\beta > 0$ sufficiently large and $|z|$ small, $f \in C^\infty(\mathbb{T}^2)$ be such that $\int_{\mathbb{T}^2} f = 0$, and for each $N > 0$, $f_N(x)$ be the discretisation of f on Λ_N. Then

$$\lim_{N\to\infty} \log \mathbb{E}^{\mathsf{SG}}_{\Lambda_N,\beta}[e^{(f_N,\phi)_{\mathbb{T}^2}}] = \underbrace{\frac{\beta(1+\mathbf{s}^{\mathsf{SG}}(\beta,z))^{-1}}{2}(f,(-\Delta_{\mathbb{T}^2})^{-1}f)_{\mathbb{T}^2}}_{\text{GFF with covariance } \beta(1+\mathbf{s}^{\mathsf{SG}})(-\Delta)^{-1}}$$

for some $\mathbf{s}^{\mathsf{SG}}(\beta,z) = O(|z|)$.

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- Expecting this :

$$\langle e^{(f,\phi)}\rangle^{\mathrm{SG}}_{\Lambda,\beta} \propto \int_{\Omega_N^{\mathrm{SG}}} d\phi\, e^{-\frac{1}{2}(\phi,-\Delta\phi)+\sqrt{\beta}(f,\phi)+z\sum_{x\in\Lambda_N}\cos(\beta^{1/2}\phi(x))}$$

$$= \int_{\Omega_N^{\mathrm{SG}}} d\phi\, \underbrace{e^{-\frac{1+\mathbf{s}^{\mathrm{SG}}}{2}(\phi,-\Delta\phi)+\sqrt{\beta}(f,\phi)}}_{\text{mgf of Gaussian}} \left(e^{\frac{\mathbf{s}^{\mathrm{SG}}}{2}(\phi,-\Delta\phi)+z\sum_{x\in\Lambda_N}\cos(\beta^{1/2}\phi(x))}\right)$$

RG on sine-Gordon

Preliminary steps 1

- Stiffness renormalisation

$$\langle e^{(f,\phi)} \rangle \propto \int_{\Omega_N^{\mathrm{SG}}} d\phi e^{-\frac{1+\mathbf{s}^{\mathrm{SG}}}{2}(\phi,-\Delta\phi)+\sqrt{\beta}(f,\phi)} e^{\frac{\mathbf{s}^{\mathrm{SG}}}{2}(\phi,-\Delta\phi)+z\sum_{x\in\Lambda_N}\cos(\beta^{1/2}\phi(x))}$$

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(rescaling $\varphi = \sqrt{1+\mathsf{s}^{\mathrm{SG}}}\phi,\ \beta' = \beta/(1+\mathsf{s}^{\mathrm{SG}}),\ s_0 = \mathsf{s}^{\mathrm{SG}}/(1+\mathsf{s}^{\mathrm{SG}})$)

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$$\propto \int_{\Omega_N^{\mathrm{SG}}} d\varphi e^{-\frac{1}{2}(\varphi,-\Delta\varphi)+\sqrt{\beta'}(f,\varphi)} \underbrace{e^{\frac{s_0}{2}(\varphi,-\Delta\varphi)+z\,\mathrm{COS}_0(\Lambda_N,(\beta')^{1/2}\varphi)}}_{Z_0(\varphi)}$$

(denoted $\mathrm{COS}_j(\Lambda_N,(\beta')^{1/2}\varphi) = L^{-2j}\sum_{x\in\Lambda_N}\cos((\beta')^{1/2}\varphi(x))$).

RG on sine-Gordon

Preliminary steps 2

- Moment generating function : let $\sum_x f(x) = 0$ (so $f \in \mathrm{Im}(-\Delta)$)

$$\langle e^{(f,\phi)} \rangle^{\mathrm{SG}}_{\Lambda,\beta} \propto \int_{\Omega_N^{\mathrm{SG}}} d\varphi e^{-\frac{1}{2}(\varphi,-\Delta\varphi)+\sqrt{\beta'}(f,\varphi)} Z_0(\varphi)$$

(Completion of square, $g = \beta^{-1/2}(-\Delta)^{-1} f$)

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(Completion of square, $g = \beta^{-1/2}(-\Delta)^{-1} f$)

$$= e^{\frac{\beta'}{2}(f,(-\Delta)^{-1}f)} \int_{\Omega_N^{\mathrm{SG}}} d\varphi e^{-\frac{1}{2}\left(\varphi-g,-\Delta(\varphi-g)\right)} Z_0(\varphi)$$

$$\propto e^{\frac{\beta'}{2}(f,(-\Delta)^{-1}f)} \mathbb{E}^{\varphi}[Z_0(\varphi+g)]$$

where $\varphi \sim \mathcal{N}(0,(-\Delta)^{-1})$, $\beta' = \beta/(1+\mathsf{s}^{\mathrm{SG}})$.

Theorem 4

Let $\beta > 0$ sufficiently large, $f \in C^\infty(\mathbb{T}^2)$ be such that $\int_{\mathbb{T}^2} f = 0$, and for each $N > 0$, $f_N(x)$ be the discretisation of f on Λ_N. Then

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for some $\mathbf{s}^{\mathrm{SG}} = O(|z|)$.

- Have : $\langle e^{(f,\phi)}\rangle_{\Lambda_N,\beta} \propto e^{\frac{\beta'}{2}(f,(-\Delta)^{-1}f)}\mathbb{E}^{\varphi}[Z_0(\varphi+g)]$.

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for some $\mathbf{s}^{\mathrm{SG}} = O(|z|)$.

- Have : $\langle e^{(f,\phi)} \rangle_{\Lambda_N,\beta} \propto e^{\frac{\beta'}{2}(f,(-\Delta)^{-1}f)} \mathbb{E}^\varphi[Z_0(\varphi + g)]$.

Goal

$$\frac{\mathbb{E}^\varphi[Z_0(\varphi+g)]}{\mathbb{E}^\varphi[Z_0(\varphi)]} \to 1 \text{ as } N \to \infty \text{ (i.e. } \Lambda_N \to \mathbb{Z}^2).$$

RG on sine-Gordon

Goal

Controlling $\mathbb{E}^{\varphi}[Z_0(\varphi + g)]$ as $N \to \infty$

$$Z_0(\varphi) = 1 + \frac{s_0}{2}|\nabla\varphi|^2_{\Lambda_N} + z\,\mathrm{COS}_0(\Lambda_N, (\beta')^{1/2}\varphi) + |\Lambda_N| o(s_0, z)$$

RG on sine-Gordon

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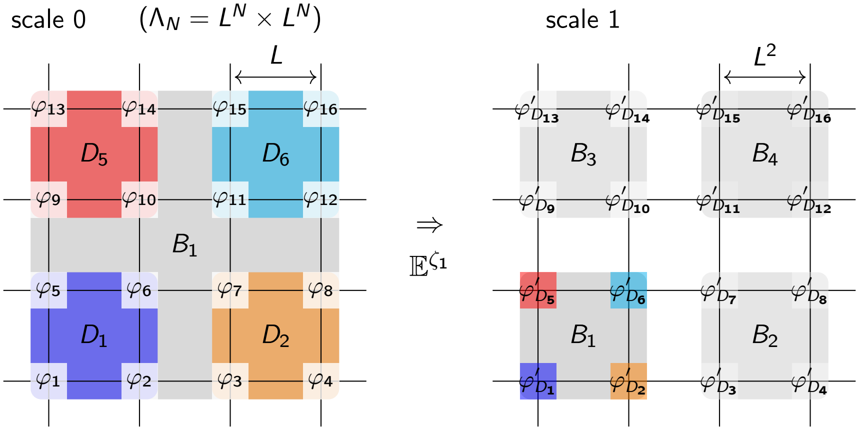
$$\Downarrow$$

$$\begin{aligned}\mathbb{E}^{\varphi}[Z_0(\varphi+g)] = 1 &+ \frac{s_0}{2}|\Lambda_N| + \frac{s_0}{2}|\nabla g|^2_{\Lambda_N} \\ &+ ze^{-\frac{\beta'}{2}(-\Delta)_{00}}L^{2N}\,\mathrm{COS}_N(\Lambda_N,(\beta')^{1/2}g) \\ &+ |\Lambda_N|\underbrace{o(s_0,z)}_{\text{not small enough}}.\end{aligned}$$

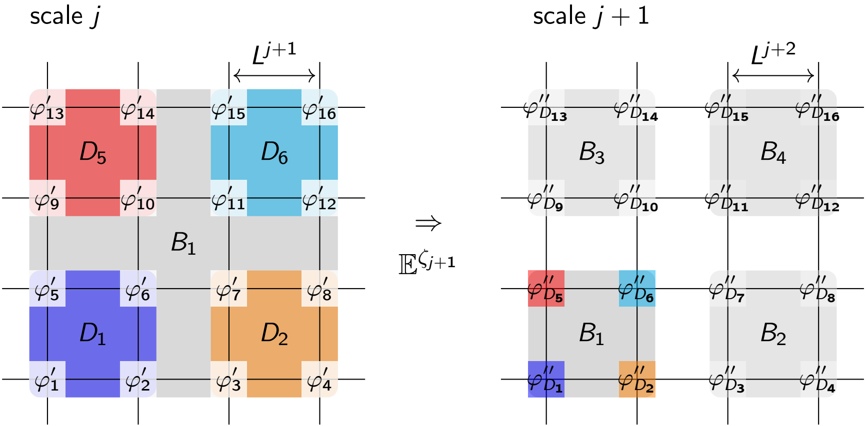
RG on sine-Gordon



(1) Field decomposition : $\varphi = \varphi' + \zeta_1$, ζ_1 consisting of fluctuations of φ in length scale $< L$.

(2) Reblocking.

RG on sine-Gordon



(1) Field decomposition : $\varphi' = \varphi'' + \zeta_{j+1}$, ζ_{j+1} consisting of fluctuations of φ' in length scale $< L^{j+1}$.

(2) Reblocking.

RG on sine-Gordon

Field / Covariance decomposition of GFF

- Want : $\varphi = \sum_{j=1}^{N} \zeta_j$ where

$$(-\Delta)^{-1} = \sum_{j=1}^{N} \Gamma_j, \quad (\zeta_j)_{1 \le j \le N} \sim \mathcal{N}(0, \bigoplus_{1 \le j \le N} \Gamma_j)$$

$$\text{s.t. } \Gamma_j(x, y) = 0 \quad \forall |x - y|_\infty > L^j.$$

called a **finite range decomposition**.

RG on sine-Gordon

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  called a **finite range decomposition**.
- A good guess :

$$(-\Delta)^{-1}(x,y) = \frac{1}{|\Lambda_N|} \sum_{k\in\Lambda_N^*} e^{ik\cdot(y-x)}(-\hat{\Delta})^{-1}(k)$$

$$\Rightarrow \quad \Gamma_j(x,y) \simeq \frac{1}{|\Lambda_N|} \sum_{k\in\Lambda_N^*} e^{ik\cdot(y-x)}(-\hat{\Delta})^{-1}(k) 1_{|k|\in[L^{-j}, L^{-j+1}]}$$

RG on sine-Gordon

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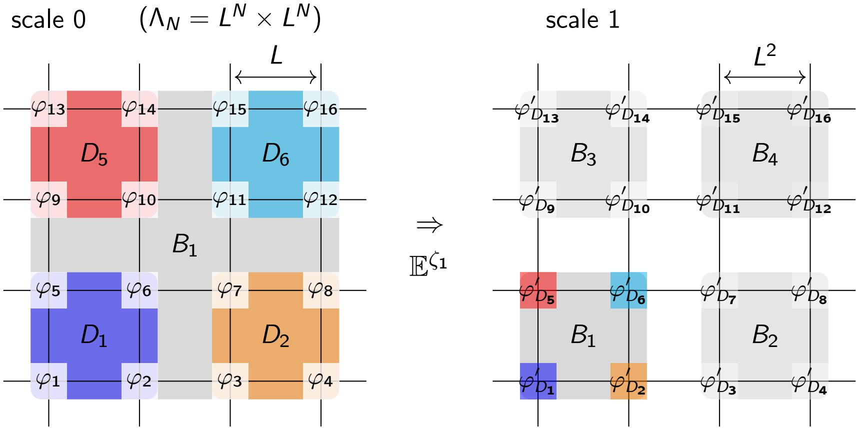
3. In limit $N \to \infty$:

$$\Gamma_j(0,0) \simeq \frac{1}{2\pi} \log L \quad \Rightarrow \quad \textbf{(fluctuations are distributed well)}$$

RG on sine-Gordon



(1) Field decomposition : $\varphi = \varphi' + \zeta_1$, ζ_1 consisting of fluctuations of φ in length scale $< L$.

(2) Reblocking.

RG on sine-Gordon

Action of $\mathbb{E}^{\zeta_1}$ on $Z_0(\varphi_1 + \zeta_1)$

$$Z_0(\varphi_1 + \zeta_1) = 1 + \frac{s_0}{2}|\nabla\varphi|^2 + z\,\mathrm{COS}_0(\Lambda_N, (\beta')^{1/2}\varphi) + |\Lambda_N| o(s_0, z).$$

- On gradient term :

$$\mathbb{E}^{\zeta_1}|\nabla(\varphi_1 + \zeta_1)|^2_{\Lambda_N} = \delta E_1 |\Lambda_N| + |\nabla\varphi_1|^2_{\Lambda_N}$$

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- On gradient term :

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- On cosine term :

$$\begin{aligned}
&\mathbb{E}^{\zeta_1}\,\mathrm{COS}_0(\Lambda_N, (\beta')^{1/2}(\varphi_1 + \zeta_1)) \\
&\quad = L^2 e^{-\frac{\beta'}{2}\Gamma_1(0,0)}\,\mathrm{COS}_1(\Lambda_N, (\beta')^{1/2}\varphi_1) \\
&\quad \simeq L^{2-\frac{\beta'}{4\pi}}\,\mathrm{COS}_1(\Lambda_N, (\beta')^{1/2}\varphi_1) \quad \left(\Gamma_1(0,0) \simeq \frac{1}{2\pi}\log L\right)
\end{aligned}$$

RG on sine-Gordon

Summary

- Field / Covariance decomposition : $\varphi = \sum_{j=1}^{N} \zeta_j$.
- Upon each $\mathbb{E}^{\zeta_{j+1}}$,

$$\begin{aligned}\log Z_1(\varphi_1) &:= \log \mathbb{E}^{\zeta_1}[Z_0(\varphi_1 + \zeta_1)] \\ &= E_1|\Lambda_N| + \frac{s_1}{2}|\nabla\varphi_1|^2 + z_1 \,\mathrm{COS}_1(\Lambda_N, (\beta')^{1/2}\varphi_1) + (\text{error})\end{aligned}$$

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RG on sine-Gordon

On the error

$$Z_j(\varphi) \propto e^{\frac{s_j}{2}|\nabla\varphi|^2 + z_j \cos_j(\Lambda_N,(\beta')^{1/2}\varphi)}\big(1 + (\text{error})\big)$$

- Ideally,

$$1 + (\text{error}) = \text{"} \prod_{B \in \text{scale j block}} \big(1 + K_j(B,\varphi)\big)\text{"}.$$

But this form is not stable under $\mathbb{E}^{\zeta_{j+1}}$.

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- Error is **parametrised by** K_j**,**

$$(\text{error}) = \sum_{X \subset \text{scale j block}} e^{-\frac{s_j}{2}(\nabla\varphi,\nabla\varphi)_X^2 - z_j \cos_j(X,(\beta')^{1/2}\varphi)} \overbrace{K_j(X,\varphi)}^{\text{polymer activities}}$$

RG on sine-Gordon

Theorem

If $\log Z_j(\varphi) = \frac{s_j}{2}|\nabla\varphi|^2 + z_j \, \mathrm{COS}_j(\Lambda_N, (\beta')^{1/2}\varphi) + F(K_j)$, then

$$
\begin{aligned}
&s_{j+1} = s_j + e_j(K_j), \quad z_{j+1} \simeq L^2 L^{-\frac{\beta'}{4\pi}} z_j, \\
&K_{j+1} = L^{-\alpha} O(K_j) + o(s_j, z_j, K_j)
\end{aligned}
$$

with initial conditions $(s_0, z_0, K_0) = (s_0, z, 0)$.

RG on sine-Gordon

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with initial conditions $(s_0, z_0, K_0) = (s_0, z, 0)$.

- If $\beta' > 8\pi$, $\exists\ s_0$ such that

$$(s_j, z_j, K_j) \xrightarrow[j\to\infty]{} (0,0,0) \quad \Leftrightarrow \quad \mathbb{E}^{\varphi}[Z_0(\varphi+g)] \sim C_N^{|\Lambda_N|}$$

Since $s_0 = \frac{\mathbf{s}^{\mathrm{SG}}}{1+\mathbf{s}^{\mathrm{SG}}}$,

$$\mathbf{s}^{\mathrm{SG}} = \frac{s_0}{1-s_0}.$$

RG on sine-Gordon

Theorem 4 (scaling limit of sine-Gordon)

Let $\beta > 0$ sufficiently large and $|z|$ small, $f \in C^\infty(\mathbb{T}^2)$ be such that $\int_{\mathbb{T}^2} f = 0$, and for each $N > 0$, $f_N(x)$ be the discretisation of f on Λ_N. Then

$$\lim_{N\to\infty} \log \mathbb{E}_{\Lambda_N,\beta}[e^{(f_N,\phi)_{\mathbb{T}^2}}] = \underbrace{\frac{\beta(1+\mathbf{s}^{\mathrm{SG}}(\beta))^{-1}}{2}(f,(-\Delta_{\mathbb{T}^2})^{-1}f)_{\mathbb{T}^2}}_{\text{GFF with covariance } \beta(1+\mathbf{s}^{\mathrm{SG}})(-\Delta)^{-1}}$$

for some $\mathbf{s}^{\mathrm{SG}} = O(|z|)$.

RG on sine-Gordon

Theorem

If $\log Z_j(\varphi) = \frac{s_j}{2}|\nabla\varphi|^2 + z_j \, \mathrm{COS}_j(\Lambda_N, (\beta')^{1/2}\varphi) + F(K_j)$, then

$$s_{j+1} = s_j + e_j(K_j), \quad z_{j+1} \simeq L^2 L^{-\frac{\beta'}{4\pi}} z_j.$$

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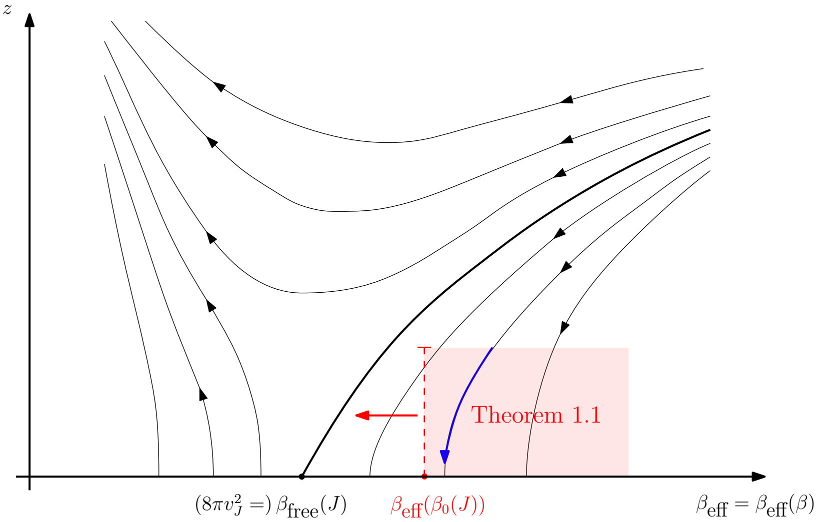
Theorem [Falco, 2012]

Using second order analysis,

$$s_{j+1} = s_j - a_j z_j^2 + e_j^1, \quad z_{j+1} \simeq L^2 L^{-\frac{\beta'}{4\pi}} (z_j - b_j s_j z_j + e_j^2),$$

i.e., satisfies the **Kosterlitz-Thouless flow equation**.

RG on sine-Gordon



Connecting sine-Gordon $\to$ dgauss

Theorem 2 [I, B.-P.-R., 2022]

"Scaling limit of dgauss is a Gaussian free field on $\mathbb{T}^2$"

Let $\beta > 0$ sufficiently large, $f \in C^\infty(\mathbb{T}^2)$ be such that $\int_{\mathbb{T}^2} f = 0$, and for each $N > 0$, $f_N(x)$ be the discretisation of f on Λ_N. Then

$$\lim_{N\to\infty} \log \mathbb{E}^{\mathrm{DG}}_{\Lambda_N,\beta}[e^{(f_N,\sigma)_{\mathbb{T}^2}}] = \frac{\beta(1+\mathsf{s}^{\mathrm{DG}}(\beta))^{-1}}{2}(f,(-\Delta_{\mathbb{T}^2})^{-1}f)_{\mathbb{T}^2}$$

with the same $\mathsf{s}^{\mathrm{DG}}(\beta) = O(e^{-c\beta})$ as in Theorem 1.

The lattice sine-Gordon model

$$Z_{N,\beta}^{\mathrm{DG}} \propto \text{"} \int_{\Omega_N^{\mathrm{SG}}} d\phi e^{-H_{N,\beta}^{\mathrm{SG}}(\phi)+(\text{"higher } q \text{ terms"})} \text{"}$$

(1) Renormalisation group method on sine-Gordon.

(2) **Why don't "higher q's" matter? Renormalisation group picture, coming back to the Discrete Gaussian model.**

Connecting sine-Gordon $\to$ dgauss

Why don't "higher q's" matter?

(2) Renormalisation group perspective.

- Recall on cosine term, $\beta' > 8\pi$:

$$\mathbb{E}^{\zeta_1}\,\mathrm{COS}_0(\Lambda_N, (\beta')^{1/2}(\varphi_1 + \zeta_1))$$
$$\simeq L^{2-\frac{\beta'}{4\pi}}\,\mathrm{COS}_1(\Lambda_N, (\beta')^{1/2}\varphi_1) \quad (\Gamma_1(0,0) \simeq \frac{1}{2\pi}\log L)$$

$$\Rightarrow \quad z_{j+1} \simeq L^{2-\frac{\beta'}{4\pi}} z_j$$

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$$\Rightarrow \quad z_{j+1} \simeq L^{2-\frac{\beta'}{4\pi}} z_j$$

- Similarly for $q \geq 2$:

$$\mathbb{E}^{\zeta_1} \mathrm{COS}_0(\Lambda_N, q(\beta')^{1/2}(\varphi_1 + \zeta_1))$$
$$\simeq L^{2-\frac{\beta'}{4\pi}q^2} \mathrm{COS}_1(\Lambda_N, q(\beta')^{1/2}\varphi_1) \quad \Leftarrow \quad \text{"irrelevant"}$$

and decays exponentially faster than $\mathrm{COS}(\Lambda_N, (\beta')^{1/2}\varphi)$.

Connecting sine-Gordon $\to$ dgauss

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with the same $\mathsf{s}^{\mathrm{DG}}(\beta) = O(e^{-c\beta})$ as in Theorem 1.

Open questions

Critical phenomena

1. GFF as a scaling limit : when $\beta = \beta_c$,

$$\lim_{\epsilon \to 0} \log \mathbb{E}^{\mathrm{DG}}_{\mathbb{Z}^2,\beta}[e^{(f_\epsilon,\sigma)}] \to \frac{\beta(1+\mathsf{s}^{\mathrm{DG}}(\beta))^{-1}}{2}(f,(-\Delta)^{-1}f)_{\mathbb{R}^2}$$

   : within reach for spread-out interaction models.

2. The behaviour of s^{DG} as $\beta \downarrow \beta_c$.

3. Decay of the correlation function when $\beta \uparrow \beta_c$.

Open questions

Observables

1. Two-point function.
2. Convergence of discrete observables to an SLE.

Related models

1. Villain model
   - Two-point observables translate into line observables.
   - Non-trivial co-homology of torus is an obstacle.
2. Non-quadratic potentials (SOS model, dual of XY model) :

$$\mathbb{P}_{\Lambda_N,\beta}(\sigma) \propto e^{-\beta^{-1}\sum_{x\sim y} V(\sigma_x-\sigma_y)}.$$

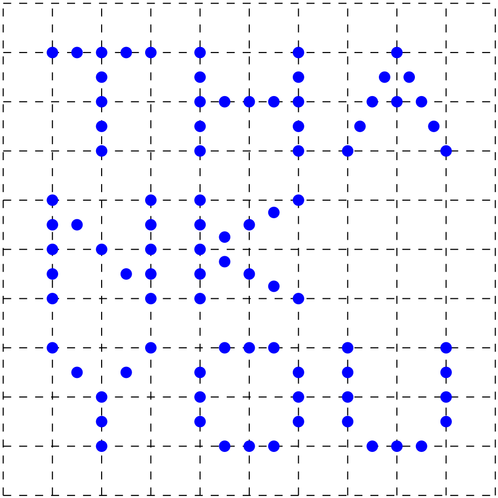

Figure 1: Lattice Thank You