Finite-size scaling of the hierarchical $|\varphi|^4$ model in $d \geq 4$

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20th April, 2024, KMS Spring meeting

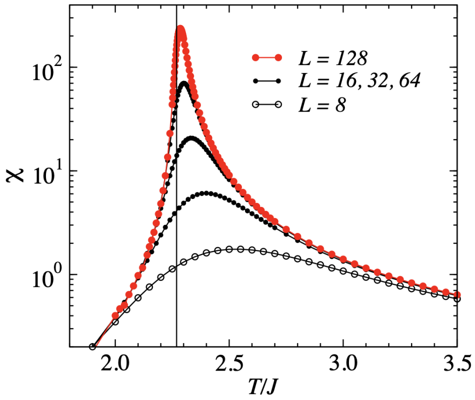
Based on the works with Gordon Slade and Emmanuel Michta

- **Boundary conditions and universal finite-size scaling for the hierarchical $|\varphi|^4$ model in dimensions 4 and higher (2023)**
- **Two-point function plateaux for the hierarchical $|\varphi|^4$ model in dimensions 4 and higher (work in progress)**

Finite-size scaling for a model of a magnet

For total magnetisation $M = \sum_x \sigma_x$,

$$\chi^{\text{tr}} = \frac{1}{\text{Vol}}\left(\langle M^2\rangle - \langle |M|\rangle^2\right) \tag{1}$$



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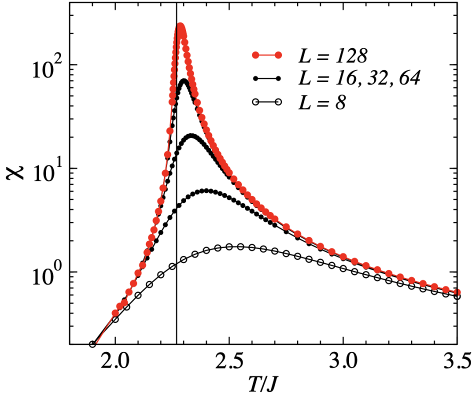


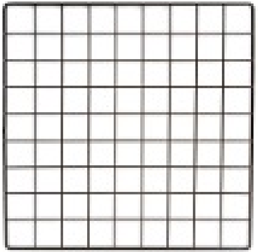
- Height of the peak?
- Width of the peak?
- Shift of the critical point?

[Sandvik, Computational Studies of Quantum Spin Systems]

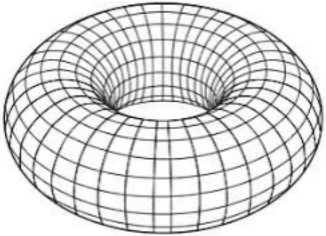
I. The $|\varphi|^4$ model

$|\varphi|^4$ model

- Lattice system $\Lambda_N = [1, L^N]^d \cap \mathbb{Z}^d$ either with free or periodic boundary condition (FBC or PBC)
- Configuration space $(\mathbb{R}^n)^{\Lambda_N} \ni \varphi$



Free BC



Periodic BC

$|\varphi|^4$ model

Lattice $|\varphi|^4$-model

For $\nu \in \mathbb{R}$, $g > 0$, the $|\varphi|^4$ model on Λ_N is given by the **Hamiltonian**

$$H_N(\varphi) = \frac{1}{2}(\varphi, (-\Delta + \nu)\varphi) + \frac{1}{4} g \sum_x |\varphi_x|^4, \qquad \varphi \in (\mathbb{R}^n)^{\Lambda_N} \tag{2}$$

and the Gibbs measure

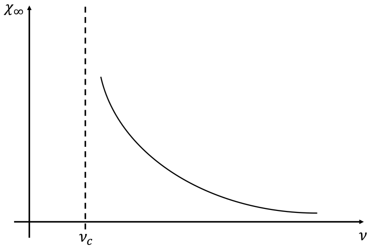
$$\mathbb{P}_{\nu,g}(d\varphi) \propto \exp\big(-H_N(\varphi)\big)\, d\varphi. \tag{3}$$

- Expectation also denoted $\langle \cdot \rangle_{\nu,g}$.

$|\varphi|^4$ model

$$H_N(\varphi) = \frac{1}{2}(\varphi, (-\Delta + \nu)\varphi) + \frac{1}{4} g \sum_x |\varphi_x|^4, \qquad \varphi \in (\mathbb{R}^n)^{\Lambda_N} \tag{4}$$

- Dimension $d \geq 4 = d_c =$ Upper critical dimension
- Susceptibility $\chi_N(\nu, g) = \sum_{x \in \Lambda_N} \langle \varphi_x \cdot \varphi_0 \rangle_{\nu,g}$.
- Critical point $\nu_c(g) = \inf\{\nu \in \mathbb{R} : \chi_\infty(\nu, g) < \infty\}$.



Known results in $d \geq 4$ at $\nu = \nu_c$

- Scaling limit
  - Gaussian limit in $d \geq 5$: macroscopic scaling limit, Fröhlich('81), Aizenman('82)
  - Gaussian limit in $d = 4$
    - Bauerschmidt, Brydges, Slade('14): ensemble scaling limit, g small with a sequence of supercritical ν approaching ν_c
    - Aizenman, Duminil-Copin('21): macroscopic scaling limit, $n = 1$

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- Correlation function
  - Gawędzki, Kupiainen('84): $d = 4$ and g small, $\langle \varphi_x \cdot \varphi_y \rangle_{g,\nu_c} \sim C_d |x - y|^{-(d-2)}$
  - Duminil-Copin, Panis('24): Ising and ϕ^4 $(n = 1)$ model with $d \geq 5$, $\langle \sigma_x \sigma_y \rangle_{\beta_c} \asymp C_d |x - y|^{-(d-2)}$

II. Motivation

II.1 Scaling limits

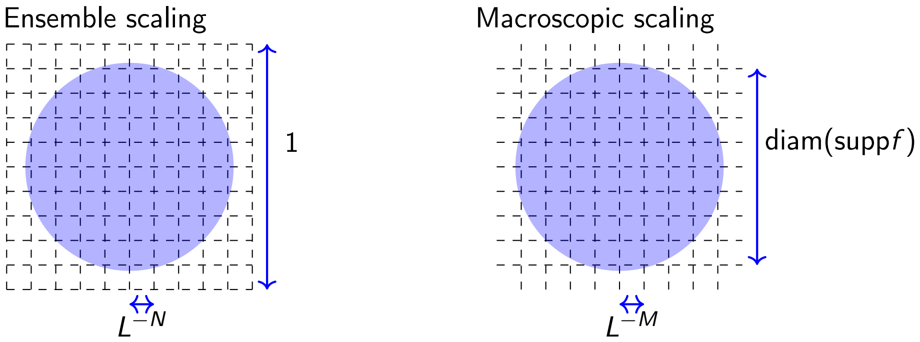
Infrared scaling limits

- Ensemble scaling limit: for $f \in C^\infty(\mathbb{T}^d)$, take $f_N(x) = f(L^{-N}x)$,

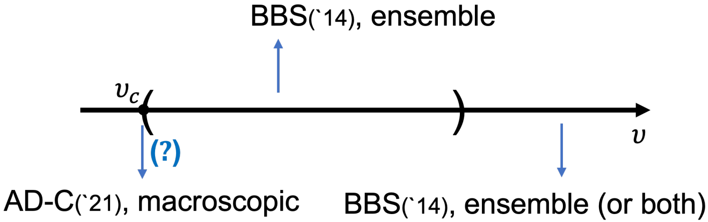
$$\lim_{N\to\infty} c_N(f_N, \varphi) =? \tag{5}$$

- Macroscopic scaling limit: for $f \in C^\infty(\mathbb{R}^d)$, take $f_M(x) = f(L^{-M}x)$,

$$\lim_{M\to\infty} \lim_{N\to\infty} c_M(f_M, \varphi) =? \tag{6}$$



Gaussian scaling limits



- How are [BBS '14] and [AD-C '21] different?
- What is the ensemble scaling limit at the critical point?

II. Motivation

II.1 Plateau

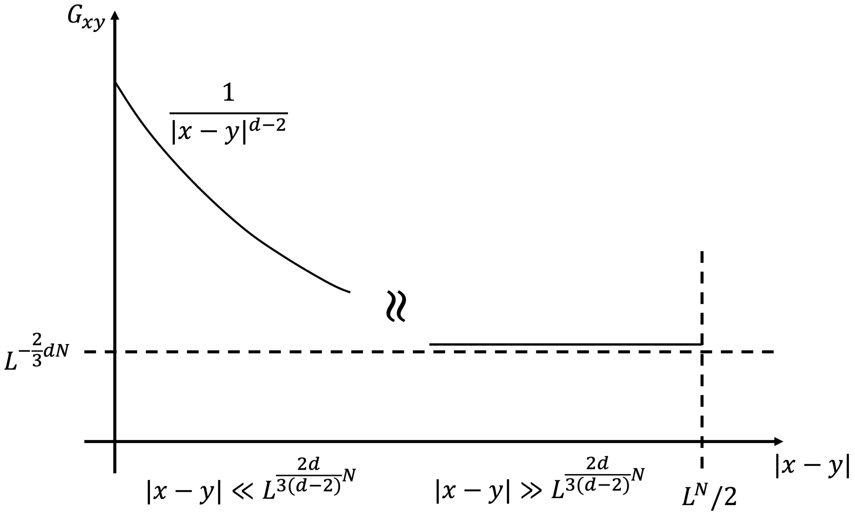
Plateau: an example

Theorem [Hutchcroft, Michta, Slade '23]

For site percolation with $d \geq 11$ in a system of size $|\Lambda| = V$,

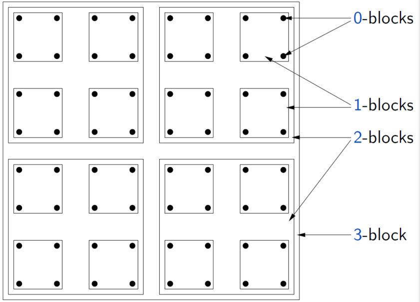
$$\mathbb{P}_{p_c,\Lambda}(0 \leftrightarrow x) \asymp \underbrace{\frac{1}{|x|^{d-2}}}_{\text{poly decay}} + \underbrace{\frac{1}{V^{\frac{2}{3}}}}_{\text{plateau}} \tag{7}$$

Plateau: an example



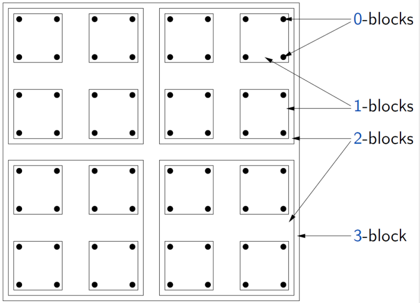
III. The hierarchical model

Hierarchical model



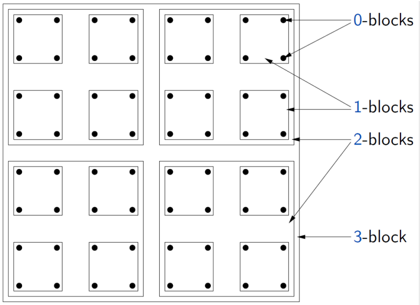
- $d_H(x, y) = L^{j_{xy}}$ where j_{xy} = smallest j s.t. x, y are in the same j-block
  - $d_H(x, y) \asymp |x - y|$ as $|x - y| \to \infty$

Hierarchical model



- $d_H(x,y) = L^{j_{xy}}$ where j_{xy} = smallest j s.t. x, y are in the same j-block
  - $d_H(x,y) \asymp |x-y|$ as $|x-y| \to \infty$
- Hierarchical RW: transition probability given by $(P_H)_{xy} = c d_H(x,y)^{-d-2}$

Hierarchical model



- $d_H(x,y) = L^{j_{xy}}$ where j_{xy} = smallest j s.t. x, y are in the same j-block
  - $d_H(x,y) \asymp |x-y|$ as $|x-y| \to \infty$
- Hierarchical RW: transition probability given by $(P_H)_{xy} = c d_H(x,y)^{-d-2}$
- **Hierarchical Laplacian** $-\Delta_H = I - P_H$

Hierarchical model

Hierarchical $|\varphi|^4$-model

For $\nu \in \mathbb{R}$, $g > 0$, the hierarchical $|\varphi|^4$ model on Λ_N is given by the **Hamiltonian**

$$H_N(\varphi) = \frac{1}{2}(\varphi, (-\Delta_H + \nu)\varphi) + \frac{1}{4}\sum_x |\varphi_x|^4, \qquad \varphi \in \mathbb{R}^{\Lambda_N} \tag{8}$$

and the probability measure

$$\mathbb{P}_{\nu,g}(d\varphi) \propto \exp\Big(- H_N(\varphi)\Big) d\varphi. \tag{9}$$

IV. Results

IV.1. Scaling limit

Critical point scaling limit

- Critical window width $w_N = L^{-\frac{d}{2}N}$ $(d > 4)$, $w_N = N^{\hat{\gamma}-\frac{1}{2}} L^{-2N}$ $(d = 4)$

Theorem (Non-Gaussian scaling and limit) [MPS '23]

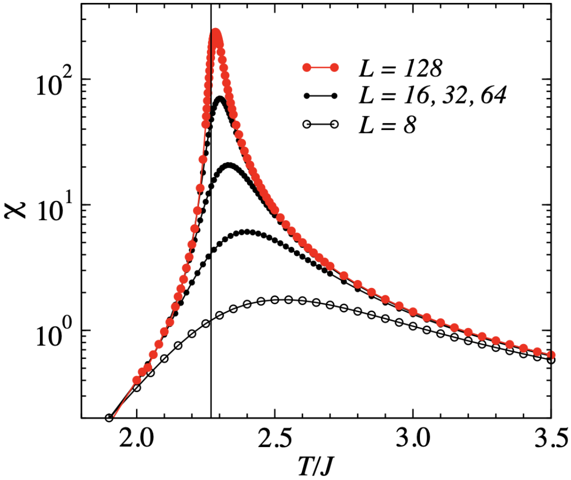
If $d \geq 4$, $n \geq 1$, $g > 0$ small and $\nu = \nu_c + s w_N$ $(s \in \mathbb{R})$, as $N \to \infty$,

$$L^{-\frac{3}{4}dN} \sum_{x \in \Lambda_N} \varphi_x \Rightarrow \text{``}e^{-\frac{1}{4}|x|^4 + cs|x|^2} dx\text{''} \quad (d > 4) \tag{10}$$

$$N^{-\frac{1}{4}} L^{-\frac{3}{4}dN} \sum_{x \in \Lambda_N} \varphi_x \Rightarrow \text{``}e^{-\frac{1}{4}|x|^4 + cs|x|^2} dx\text{''} \quad (d = 4) \tag{11}$$

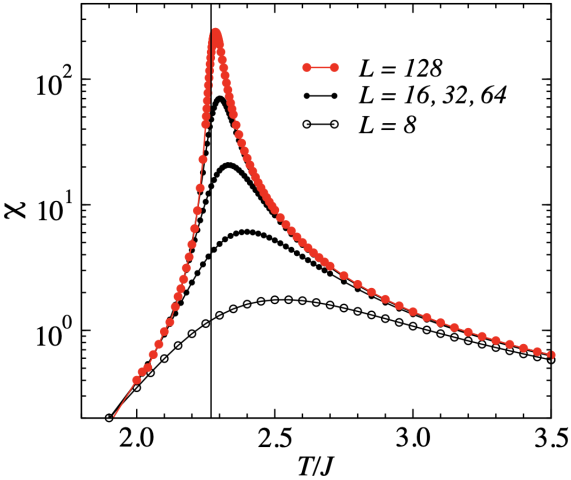
and thus

$$\chi_N(\nu_c) \sim C_d L^{\frac{d}{2}N} \times \begin{cases} 1 & (d > 4) \\ N^{1/2} & (d = 4) \end{cases} \tag{12}$$



- Height of the peak $= \chi_N(\nu_c)$

$$\chi_N(\nu_c) \sim C_d L^{\frac{d}{2}N} \times \begin{cases} 1 & (d > 4) \\ N^{1/2} & (d = 4) \end{cases}$$



- Height of the peak $= \chi_N(\nu_c)$

$$\chi_N(\nu_c) \sim C_d L^{\frac{d}{2}N} \times \begin{cases} 1 & (d > 4) \\ N^{1/2} & (d = 4) \end{cases}$$

- Width of the peak $= w_N$

$$w_N = L^{-\frac{d}{2}N} \times \begin{cases} 1 & (d > 4) \\ N^{\frac{n+2}{n+8} - \frac{1}{2}} & (d = 4) \end{cases}$$

Critical point scaling limit

Theorem [MPS '23]

$$L^{-\frac{3}{4}dN} \sum_{x\in\Lambda_N} \varphi_x \Rightarrow \text{``}e^{-\frac{1}{4}|x|^4+cs|x|^2}dx\text{''} \quad (d>4) \tag{13}$$

$$N^{-\frac{1}{4}} L^{-\frac{3}{4}dN} \sum_{x\in\Lambda_N} \varphi_x \Rightarrow \text{``}e^{-\frac{1}{4}|x|^4+cs|x|^2}dx\text{''} \quad (d=4) \tag{14}$$

Why is this an indication of non-Gaussian scaling limit?

- According to [BBS '14], $L^{-\frac{d+2}{2}N}(f,\varphi) \Rightarrow \mathcal{N}(0,(f,(-\Delta+\alpha s)^{-1}f))$
- If $f=\bar{f}+(f-\bar{f})$, since $L^{\frac{3}{4}dN} \gg L^{\frac{d+2}{2}N}$ $(d>4)$ and $N^{\frac{1}{4}}L^{\frac{3}{4}dN} \gg L^{\frac{d+2}{2}N}$ $(d=4)$, $(f-\bar{f},\varphi)$ vanishes under the scaling in the theorem
- **Thus the contribution of $(\bar{f},\varphi) = L^{-dN}\sum_x f_x \sum_x \varphi_x$ dominates!**

Critical point scaling limit, FBC

- Critical point shift $v_N \propto L^{-2N}$ $(d > 4)$, $v_N \propto L^{-2N} N^{\hat{\gamma}}$ $(d = 4)$

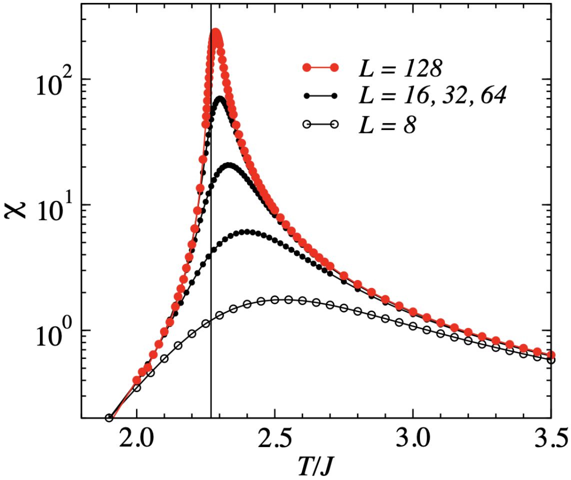
Theorem (FBC scaling) [MPS '23]

If $d \geq 4$, $n \geq 1$, $g > 0$ small and $\nu = \nu_c - v_N$, with FBC,

$$L^{-\frac{3}{4}dN} \sum_{x \in \Lambda_N} \varphi_x \Rightarrow \text{“} e^{-\frac{1}{4}|x|^4} dx \text{”} \quad (d > 4) \tag{15}$$

$$L^{-\frac{3}{4}dN} N^{-\frac{1}{4}} \sum_{x \in \Lambda_N} \varphi_x \Rightarrow \text{“} e^{-\frac{1}{4}|x|^4} dx \text{”} \quad (d = 4) \tag{16}$$

- Clarifies the difference between [BBS '14] and [AD-C '21]



- Height of the peak $= \chi_N(\nu_c)$
- Width of the peak $= w_N$
- Shift of the critical point $= v_N$

$$v_N \propto L^{-2N} \times \begin{cases} 1 & (d > 4) \\ N^{\frac{n+2}{n+8}} & (d = 4) \end{cases}$$

IV. Results

IV.2. Plateau

Plateau

Theorem [Park, Slade, work in progress]

Let $d > 4$, $g > 0$ and $\nu = \nu_c(g)$.

- If $|x_N| \to \infty$ with $|x_N| \ll L^{\frac{d}{2(d-2)}N}$, then
$$\langle \varphi_0 \varphi_{x_N} \rangle_{g,\nu} \sim \frac{1}{|x_N|^{d-2}} \tag{17}$$
- If $|x_N| \gg L^{\frac{d}{2(d-2)}N}$, then
$$\langle \varphi_0 \varphi_{x_N} \rangle_{g,\nu} \sim C(g) L^{-dN/2} \tag{18}$$

Theorem [PS, work in progress]

Let $d = 4$, $g > 0$ and $\nu = \nu_c(g)$.

- If $|x_N| \to \infty$ with $|x_N| \ll N^{-\frac{1}{4}} L^{\frac{d}{2(d-2)}N}$, then
$$\langle \varphi_0 \varphi_{x_N} \rangle_{g,\nu} \sim \frac{1}{|x_N|^{d-2}} \tag{19}$$
- If $|x_N| \gg N^{-\frac{1}{4}} L^{\frac{d}{2(d-2)}N}$, then
$$\langle \varphi_0 \varphi_{x_N} \rangle_{g,\nu} \sim C(g) N^{1/2} L^{-dN/2} \tag{20}$$

Plateau

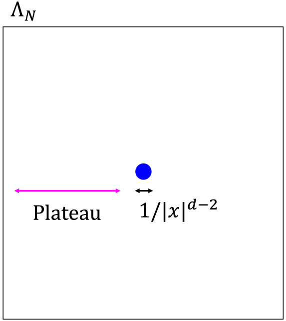
Theorem [PS, work in progress, $d > 4$]

- If $|x_N| \to \infty$ with $|x_N| \ll L^{\frac{d}{2(d-2)}N}$, then
$$\langle \varphi_0 \varphi_{x_N} \rangle_{g,\nu} \sim \frac{1}{|x_N|^{d-2}} \tag{21}$$
- If $|x_N| \gg L^{\frac{d}{2(d-2)}N}$, then
$$\langle \varphi_0 \varphi_{x_N} \rangle_{g,\nu} \sim C(g) L^{-dN/2} \tag{22}$$

Shows a plateau:

(1) $L^{\frac{d}{2(d-2)}N} \ll L^N$

(2) $C(g) L^{-dN/2}$ is a constant



Conjectures/Prospectives

Related results:

- Plateau phenomenon (dealt in a work in progress)
- Limiting distribution at $\nu = \nu_c$ of (f, φ) (both for $\bar{f} \neq 0$ and $= 0$)

The same picture would hold models in the same universality class:

1. Euclidean (usual) $|\varphi|^4$-model in $d \geq 4$
2. $O(n)$ lattice spin models in $d \geq 4$ ($n = 1$: Ising model, $n = 2$: XY model, $n = 3$: Heisenberg model)
3. (Strictly or weakly) Self-avoiding walks

Thank you