Finite-size scalings of the Euclidean $|\varphi|^4$ model at and above the critical dimension

Jiwoon Park

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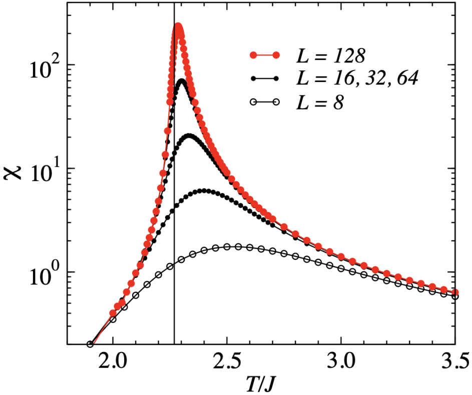
- P., Torus scaling limits and the plateau of the critical weakly coupled $|\varphi|^4$ model in $d \geq 4$, *arXiv:2511.06321* (2025)
- P., A Rernomalisation Group Map for Short- and Long-ranged Weakly Coupled $|\varphi|^4$ Models in $d \geq 4$ at and Above the Critical Point, *arXiv:2511.03495* (2025)
- E. Michta, P., G. Slade, Boundary conditions and universal finite-size scaling for the hierarchical $|\varphi|^4$ model in dimensions 4 and higher, *CPAM* (2023)
- P., G. Slade, Two-point function plateaux for the hierarchical $|\varphi|^4$ model in dimensions 4 and higher, *AHP* (2024)
- Y. Liu, P., G. Slade, Universal finite-size scaling in high-dimensional critical phenomena, *arXiv:2412.08814* (2024)

Finite-size scaling for a model of a magnet

For total magnetisation $M = \sum_x \sigma_x$,

$$\chi^{\text{tr}} = \frac{1}{\text{Vol}}\left(\langle M^2\rangle - \langle |M|\rangle^2\right)$$



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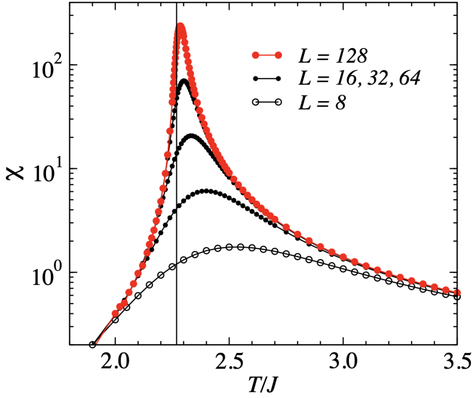


- Height of the peak?
- Width of the peak?
- Shift of the critical point?

[Sandvik, Computational Studies of Quantum Spin Systems]

Definitions and motivations

$|\phi|^4$ model

- $\Lambda_N = \{1, \cdots, L^N\}^d$: d-dimensional lattice box (with periodic boundary condition)
- $\Omega_N = \{\phi : \Lambda_N \to \mathbb{R}^n\}$: configuration space
- Coupling constants $g > 0$, $\nu \in \mathbb{R}$

Definition

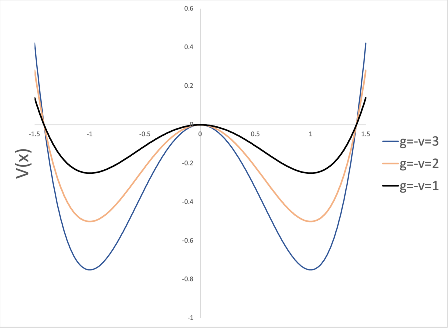
The $|\phi|^4$ **model** with coupling constants g, ν is the probability measure on Ω_N given by

$$\langle F(\phi)\rangle_{g,\nu,N} = \frac{1}{Z_{N,g,\nu}} \int_{\Omega_N} d\phi\, F(\phi) \exp\left(-\tfrac{1}{2}(\phi, -\Delta\phi) - \tfrac{1}{2}\nu \sum_x |\phi_x|^2 - \tfrac{1}{4} g \sum_x |\phi_x|^4\right).$$

- $d \geq 4 = d_c$, the upper critical dimension
- When $g \ll 1$, then a weakly-coupled $|\phi|^4$-model

$|\phi|^4$ model

$$\langle F(\phi)\rangle_{g,\nu,N} = \frac{\int_{\Omega_N} d\phi F(\phi) e^{-\frac{1}{2}(\phi,-\Delta\phi)} e^{-\sum_x V_x(\phi)}}{Z_{g,\nu,N}}, \qquad V_x(\phi) = \frac{\nu}{2}|\phi_x|^2 + \frac{g}{4}|\phi_x|^4$$



- Under the limit $g = -\nu \to \infty$, converges to the $O(n)$ model

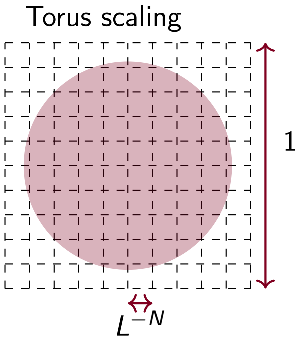
Infrared scaling limits

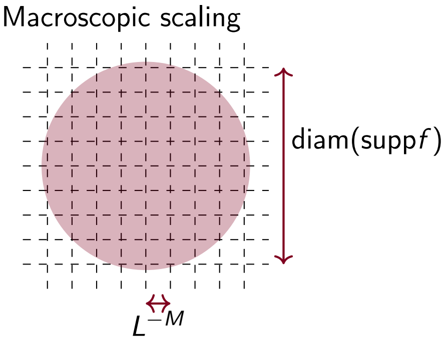
- Torus scaling limit: for $f \in C^\infty(\mathbb{T}^d)$, take $f_N(x) = f(L^{-N}x)$,

$$\lim_{N\to\infty} c_N(f_N, \phi) =?$$

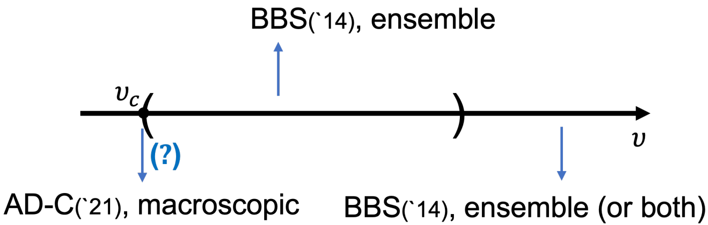
- Macroscopic scaling limit: for $f \in C^\infty(\mathbb{R}^d)$, take $f_M(x) = f(L^{-M}x)$,

$$\lim_{M\to\infty} \lim_{N\to\infty} c_M(f_M, \phi) =?$$





Gaussian scaling limits



- How are [BBS '14] and [AD-C '21] different?
- What is the torus scaling limit at the critical point?

Plateau: examples

Theorem [Liu, Panis, Slade '24]

For the Ising model in $d > 4$ in a system of size $|\Lambda| = V$ and $\beta_* = \beta_c - c|\Lambda|^{-1/2}$,

$$\langle \sigma_0 \sigma_x \rangle_{\beta_*,\Lambda} \asymp \underbrace{\frac{1}{|x|^{d-2}}}_{\text{poly decay}} + \underbrace{\frac{1}{V^{\frac{1}{2}}}}_{\text{plateau}}$$

Theorem [Liu, Slade '24]

For the lattice trees and animals in $d > 8$ in a system of size $|\Lambda| = V$ and $p_* = p_c - c|\Lambda|^{-1/2}$,

$$\mathbb{P}_{p_*,\Lambda}(0 \leftrightarrow x) \asymp \frac{1}{|x|^{d-2}} + \frac{1}{V^{\frac{3}{4}}}$$

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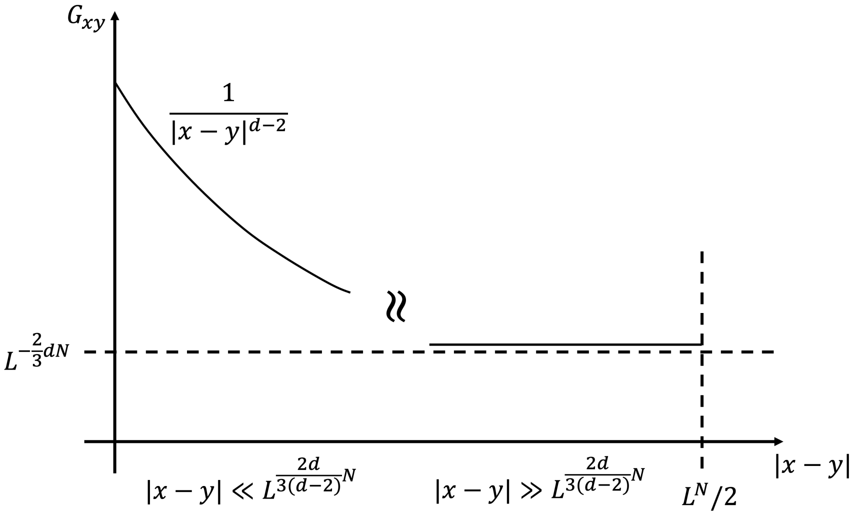
Theorem [Hutchcroft, Michta, Slade '23]

For the Bernoulli percolation with $d \geq 11$ in a system of size $|\Lambda| = V$,

$$\mathbb{P}_{p_c,\Lambda}(0 \leftrightarrow x) \asymp \frac{1}{|x|^{d-2}} + \frac{1}{V^{\frac{2}{3}}}$$

Plateau: Bernoulli percolation



Main results

Critical point of the $|\phi|^4$ model, $d \geq 4$

- Natural injection $i_N : \Lambda_N \to \mathbb{T}^d$
- $f \in C^\infty(\mathbb{T}^d; \mathbb{R}^n)$, let $f_N : \Lambda_N \to \mathbb{R}^n$ be the discretisation given $f_N(x) = f(i_N(x))/|\Lambda_N|$
- $\bar{f} = \int_{\mathbb{T}^d} f(x)dx$, $\bar{f}_N = \frac{1}{|\Lambda_N|} \sum_x f_N(x)$

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Theorem (BBS'14 for $d = 4$, P'25$^+$ for $d \geq 5$)

Let $d \geq 4$, g be sufficiently small. Then there exist $\nu_c \equiv \nu_c(g)$ such that, for some sequence $(\epsilon_k, \epsilon'_k) \to (0,0)$,

$$\lim_{N\to\infty} \left\langle e^{(\phi, f_N)/L^{Nd/2}} \right\rangle_{g,\nu_c+\epsilon_k,N} := \mathbf{WN}\left[e^{(\epsilon'_k)^{-1/2}(\psi,f)}\right] = \exp\left(\frac{1}{2\epsilon'_k}(f,f)\right).$$

- **WN** is the White noise measure on $\mathbb{T}^d$, i.e., absence of long range order

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The finite-volume susceptibility is defined as $\chi_{g,\nu,N} = \frac{1}{|\Lambda_N|}\langle (\sum_x \phi_x)^2 \rangle_{g,\nu,N}$.

Corollary

Under the same assumptions, $\chi_{g,\nu,\infty} = \lim_{N\to\infty} \chi_{g,\nu,N}$ exists and

$$\chi_{g,\nu_c+\epsilon_k,\infty} = \frac{1}{\epsilon'_k} \to \infty \qquad \text{as} \qquad \epsilon_k \to 0.$$

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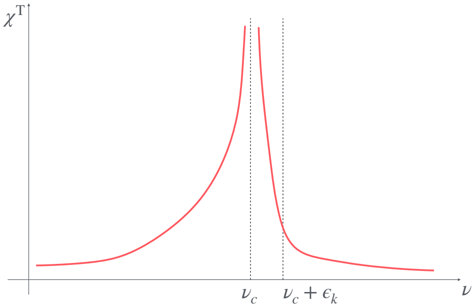

Torus scaling limit of $|\phi|^4$ model, $d \geq 5$

Theorem (P'25$^+$)

Let $d \geq 5$, g be sufficiently small, $b_N = g^{-1/4} L^{-dN/4}$ and $c_N = L^{-\frac{d-2}{2}N}$.

1. *There exists $\gamma = 1 + O(g) > 0$ such that*

$$\lim_{N\to\infty} \left\langle e^{(\phi, f_N)/b_N} \right\rangle_{g,\nu_c,N} = \mathbf{Q}\left[e^{\gamma(\psi,f)}\right] \propto \int e^{\gamma y\cdot\bar{f}} e^{-|y|^4} dy.$$

2. *There exists $\beta = 1 + O(g) > 0$ such that*

$$\lim_{N\to\infty} \left\langle e^{(\phi,(f_N - \bar{f}_N))/c_N} \right\rangle_{g,\nu_c,N} = \mathbf{GFF}_0\left[e^{\beta(\psi, f-\bar{f})}\right].$$

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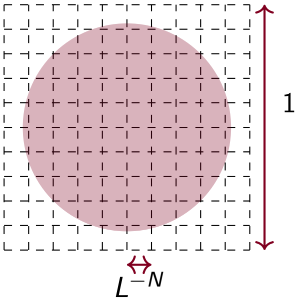
Torus scaling



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Interpretation

- If $Y \sim$ "$e^{-|y|^4}$", then $Y\mathbf{1} \sim \mathbf{Q}$.
- Thus on a finite-volume torus,

$$\phi \text{ “}\sim\text{” } \gamma b_N Y\mathbf{1} + \beta c_N \mathrm{GFF},$$

- Since $b_N \gg c_N$, we see that there is a scale hierarchy where ϕ is constant on the scale of the torus and the GFF on microscopic scales.

Torus scaling limit of $|\phi|^4$ model, $d = 4$

Theorem (P'25$^+$)

Let $d = 4$, g be sufficiently small, $b_N = N^{1/4} L^{-N}$ and $c_N = L^{-N}$.

1. *There exists $\gamma > 0$, independent of g, such that*

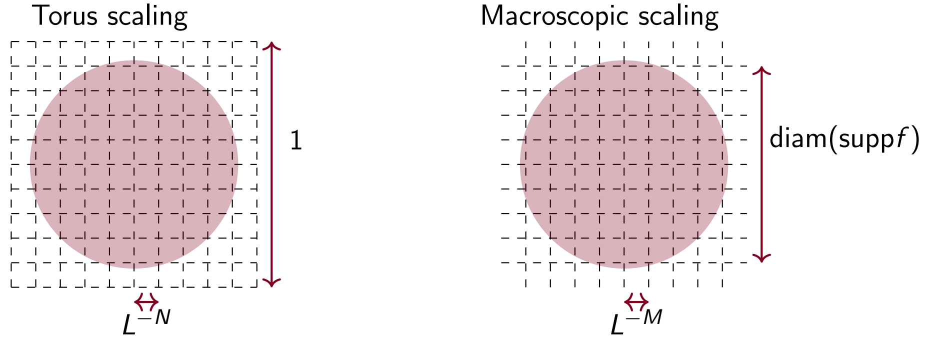
$$\lim_{N\to\infty} \left\langle e^{(\phi,f_N)/b_N} \right\rangle_{g,\nu_c,N} = \mathbf{Q}\left[e^{\gamma(\psi,f)}\right] \propto \int e^{\gamma y\cdot \bar{f}} e^{-|y|^4} dy.$$

2. *There exists $\beta = 1 + O(g) > 0$ such that*

$$\lim_{N\to\infty} \left\langle e^{(\phi,(f_N-\bar{f}_N))/c_N} \right\rangle_{g,\nu_c,N} = \mathbf{GFF}_0 \left[e^{\beta(\psi,f-\bar{f})}\right].$$

Comparison with previous results

- Macroscopic scaling limit
  - Gaussian limit in $d \geq 4$ and $n = 1$: Aizenman, Duminil-Copin('21), Fröhlich('81), Aizenman('82)
- Torus scaling limit
  - Weakly coupled, any $n \geq 1$
  - Scale hierarchy ϕ " $\sim$ " $b_N Y \mathbf{1} + c_N \mathrm{GFF}$
  - Competition between the 'plateau effect' and the Gaussian limit



Comparison with previous results

With $b_N = N^{1/4}L^{-N}$ for $d=4$, $b_N = g^{-1/4}L^{-dN/4}$ for $d>4$ $c_N = L^{-\frac{d-2}{2}N}$), suppose

$$\psi \text{ “ = ” } \gamma b_N Y \mathbf{1} + \beta c_N \mathrm{GFF}$$

Corollary (Macroscopic limit for ψ)

For $f \in C_c^\infty(\mathbb{R}^d)$, let $f^M(x) = f(L^M x)$. Then

$$\lim_{N\to\infty}\lim_{M\to\infty}\left\langle e^{L^M(\psi, f_N^M)/c_N}\right\rangle = \exp\left(\frac{1}{2}\beta(f, (-\Delta)^{-1}f)\right).$$

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- Gives an covariance structure of the macroscopic scaling limit
- We did not prove that $\phi \text{ ``} \sim \text{''} \gamma b_N Y\mathbf{1} + \beta c_N \mathrm{GFF}$ is uniform in M, but expected to be true.

Plateau

Theorem [Park, 25$^+$]

Let $d > 4$, $g > 0$ and $\nu = \nu_c(g)$.

- If $|x_N| \to \infty$ with $|x_N| \ll L^{\frac{d}{2(d-2)}N}$, then
$$\langle \phi_0 \phi_{x_N} \rangle_{g,\nu} \sim \frac{c_1}{|x_N|^{d-2}}$$
- If $|x_N| \gg L^{\frac{d}{2(d-2)}N}$, then
$$\langle \phi_0 \phi_{x_N} \rangle_{g,\nu} \sim c_2 g^{-1/4} L^{-dN/2}$$

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Plateau

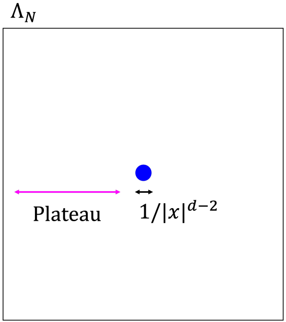
Theorem [Park, 25^+, $d > 4$]

- If $|x_N| \to \infty$ with $|x_N| \ll L^{\frac{d}{2(d-2)}N}$, then
$$\langle \phi_0 \phi_{x_N} \rangle_{g,\nu} \sim \frac{c_1}{|x_N|^{d-2}}$$
- If $|x_N| \gg L^{\frac{d}{2(d-2)}N}$, then
$$\langle \phi_0 \phi_{x_N} \rangle_{g,\nu} \sim c_2 g^{-1/4} L^{-dN/2}$$

Shows a plateau:

(1) $L^{\frac{d}{2(d-2)}N} \ll L^N$

(2) $c_2 g^{-1/4} L^{-dN/2}$ is a constant



Conjectures/Prospectives

Related results:

- Presence of the critical window and a scaling profile

The same picture would hold models in the same universality class:

1. $O(n)$ lattice spin models in $d \geq 4$ ($n = 1$: Ising model, $n = 2$: XY model, $n = 3$: Heisenberg model)
2. (Strictly or weakly) Self-avoiding walks

There is a problem with the boundary condition:

- Occurrence of the pseudocritical point under FBC
- Appearance of the same scaling profile about the pseudocritical point, just with different constants

Case of convex potentials

Spin systems with convex potential

- $\Lambda_N = \{1, \cdots, L^N\}^d$: d-dimensional lattice box (with periodic boundary condition)
- $\Omega_N = \{\varphi : \Lambda_N \to \mathbb{R}^n\}$: configuration space (For convenience, $n = 1$)
- Convex function $W : \mathbb{R}^n \to \mathbb{R}$

Definition

We define the spin system with potential W as a probability measure on Ω_N given by

$$\langle F(\phi) \rangle_{W,N} = \frac{1}{Z_{W,N}} \int_{\Omega_N} d\phi \, F(\phi) \exp\left(-\tfrac{1}{2}(\phi, -\Delta\phi) - W(\varphi)\right) \prod_x d\phi, \qquad \phi \in \Omega_N.$$

Methods for convex potentials

A corollary of the Prèkopa–Leindler inequality

If $D^2 S \geq 0$ and μ is some probability measure, then $\mu * e^{-S}$ is log-concave, i.e., if $\zeta \sim \mu$, then

$$\varphi \mapsto -\log \mu[e^{-S(\varphi+\zeta)}], \qquad \varphi \in \Omega_N$$

is a convex function.

Brascamp-Lieb inequality

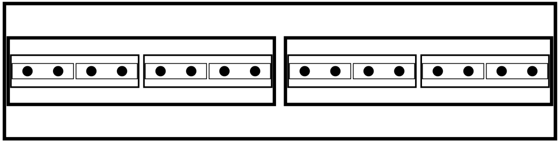
If W is even and $D^2 W(\phi) \geq Q > 0$ for a quadratic form Q, uniformly in ϕ, then for any $h \in \Omega_N$,

$$\langle e^{(h,\phi)} \rangle_{W,N} \leq \exp\left(\frac{1}{2}(h, (-\Delta + Q)^{-1} h) \right)$$

- In particular, if $g, \nu > 0$, then the BL inequality implies $\langle \phi_0 \phi_x \rangle_{g,\nu,\infty} \leq O(e^{-\mu|x|})$.
- More detailed information can be obtained by Helffer-Sjörstrand representation

RG method

Idea of renormalisation group



CLT and RG:

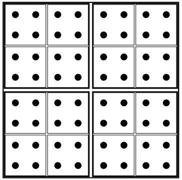
- Let $(X_n)_{n\geq 0}$ be i.i.d., $\mathbb{E}[X_n^2] = 1$, $\mathbb{E}[X_n] = 0$, $X_n \sim e^{-v(x)}/Z_v$, considered as 'spin chain'
- Let $Y_{j,n} = (X_{n2^j+1} + \cdots + X_{(n+1)2^j})/2^j$, then

$$Y_{j,n} \sim e^{-v_j(x)}/Z_j = (e^{-v}/Z_v)^{*2^j}$$

- According to the CLT, $v_j(x) \to \frac{1}{2}x^2$ as $j \to \infty$, so averaged field improves convexity.
- Reflecting on the proof of the CLT, this process can be shown inductively.

Reformulation using Gaussian integral

$$\langle F(\phi)\rangle_{g,\nu,N} = \frac{\int_{\Omega_N} d\phi F(\phi) e^{-\frac{1}{2}(\phi,-\Delta\phi)} e^{-\sum_x V_x(\phi)}}{Z_{g,\nu,N}}, \quad V_x(\phi) = \frac{\nu}{2}|\phi_x|^2 + \frac{g}{4}|\phi_x|^4$$



$\mathcal{B}_j$ = set of boxes of L^j points = set of j-boxes

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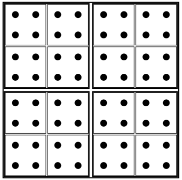

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- Step 1: Regroup lattice field $\phi(B) = (\phi_x)_{x\in B}$ for each block $B \in \mathcal{B}_j$, compute the law of each $\phi(B)$

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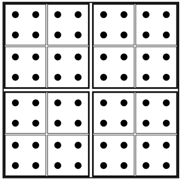

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- Step 1: Regroup lattice field $\phi(B) = (\phi_x)_{x\in B}$ for each block $B \in \mathcal{B}_j$, compute the law of each $\phi(B)$
- Step 2: Compute the j-scale interaction of $(\phi(B))_{B\in\mathcal{B}_j}$ that replaces $(\phi, -\Delta\phi)$

Reformulation using Gaussian integral

$$\phi \sim \mathcal{N}(0, (-\Delta + \mu^2)^{-1}) \qquad \Rightarrow \qquad Z_{g,\nu,N} \propto \lim_{\mu \to 0} \mathbb{E}^{\phi}[e^{-\sum_x V_x(\phi)}]$$

Lemma

Suppose $\zeta_1 \sim \mathcal{N}(0, C_1)$, $\zeta_2 \sim \mathcal{N}(0, C_2)$ *are independent. Then* $\zeta_1 + \zeta_2 \sim \mathcal{N}(0, C_1 + C_2)$.

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- Suppose there exist $\Gamma_1, \cdots, \Gamma_N$ such that

$$(-\Delta + \mu^2)^{-1} = \sum_{j=1}^{N} \Gamma_j + \mu^{-2} Q_N$$

  and $\zeta_j \sim \mathcal{N}(0, \Gamma_j)$ is the 'scale j fluctuation' of the massive GFF. Let $\hat{\zeta} \sim \mathcal{N}(0, \mu^{-2} Q_N)$.

Reformulation using Gaussian integral

$$\phi \sim \mathcal{N}(0, (-\Delta + \mu^2)^{-1}) \qquad \Rightarrow \qquad Z_{g,\nu,N} \propto \lim_{\mu \to 0} \mathbb{E}^{\phi}[e^{-\sum_x V_x(\phi)}]$$

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- By the lemma, $\zeta_1 + \cdots + \zeta_N + \hat{\zeta} =^d \phi$, so

$$Z_{g,\nu,N} \propto \lim_{\mu \to 0} \mathbb{E}^{\hat{\zeta}} \mathbb{E}^{\zeta_N} \cdots \mathbb{E}^{\zeta_1} \Big[\exp(-\sum_x V_x(\zeta_1 + \cdots + \zeta_N + \hat{\zeta})) \Big].$$

Renormalised potential

Given $(\zeta_j)_{j=1}^N \sim \mathcal{N}(0, \Gamma_1 \oplus \cdots \oplus \Gamma_N)$ with $\sum_j \Gamma_j = (-\Delta + \mu^2)^{-1}$,

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- For each j, the renormalised potential is defined as a partial expectation

$$e^{-U_j(\varphi)} = \mathbb{E}^{\zeta_j} \cdots \mathbb{E}^{\zeta_1} \Big[\exp(-\sum_x V_x(\varphi + \zeta_1 + \cdots + \zeta_j)) \Big]$$

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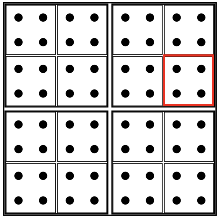
$$\Rightarrow \qquad Z_{g,\nu,N} = \lim_{\mu \to 0} \mathbb{E}^{\hat{\zeta}} \underbrace{\mathbb{E}^{\zeta_N} \cdots \mathbb{E}^{\zeta_{j+1}}}_{\text{scale } j \text{ interaction}} \Big[\underbrace{\exp(-U_j(\zeta_{j+1} + \cdots + \zeta_N + \hat{\zeta}))}_{\text{scale } j \text{ potential function}} \Big]$$

Renormalised potential

$$e^{-U_j(\phi)} = \mathbb{E}^{\zeta_j} \cdots \mathbb{E}^{\zeta_1}\Big[\exp(-\sum_x V_x(\phi + \zeta_1 + \cdots + \zeta_j))\Big]$$



$\phi(B')$

Interaction $\Gamma_{j+1} + \cdots + \Gamma_N$

$\phi(B)$

Renormalised potential

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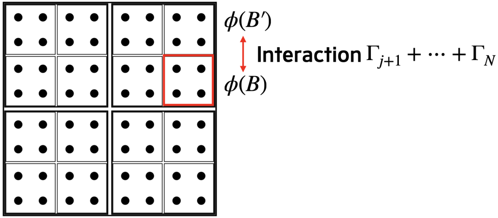

Goal: Approximate U_j using 'local' polynomials

$$U_j(\phi) = u_j |\Lambda_N| + \sum_x \frac{1}{2}\nu_j |\phi_x|^2 + \frac{1}{4} g_j |\phi_x|^4 + \cdots$$

and analyse the stability of the dynamical system $(u_j, \nu_j, g_j)_{j \geq 0}$.

Renormalised potential

Theorem

If $\nu = \nu_c$, there exists vacuum energy u_j, a 'local' polynomial $V_j(\varphi)$ and a collection of smooth function $(K_j(X) : X \subset \mathcal{B}_j)$ such that

$$e^{-U_j(\phi)} = e^{-u_j|\Lambda|} e^{-\sum_x V_{j,x}(\phi)} \mathcal{E}xp(K_j)(\phi)$$

and

$$\|V_j\| = O(\epsilon_j), \qquad \|K_j\| = O(\epsilon_j^3)$$

for some $\epsilon_j \to 0$ and suitable norms on function spaces.

- V_j is a low order approximation of U_j as a local polynomial
- K_j is a remainder term and $\mathcal{E}xp$ is a cluster expansion of scale j polymers given by

$$\mathcal{E}xp(K) = 1 + \sum_{X \subset \mathcal{B}_j,\, X \neq \emptyset} K_j(X)$$

Finite-size susceptibility and the effective potential

Finite-size susceptibility can be computed as the following:

- $Z_{g,\nu,N} \propto \lim_{\mu\to 0} \mathbb{E}^{\hat{\zeta}}[e^{-U_N}]$ with $\hat{\zeta} \sim \mathcal{N}(0, \mu^2 Q_N)$, and $\hat{\zeta} = Y'\mathbf{1}$ for $Y' \sim \mathcal{N}(0, \mu^2 L^{-dN})$,

$$\begin{aligned} Z_{g,\nu_c,N} &\propto \lim_{\mu\to 0} \int_{\mathbb{R}^n} (e^{-L^{dN} V_N(y)} + K_N(\Lambda_N, y\mathbf{1}))e^{-\frac{L^{dN}}{2\mu^2}|y|^2} dy \\ &= \lim_{\mu\to 0} \int_{\mathbb{R}^n} \left(e^{-\frac{1}{4}L^{dN} g_N |y|^4 - \frac{1}{2} L^{dN} \nu_N |y|^2} + K_N(\Lambda_N, y\mathbf{1})\right) e^{-\frac{L^{dN}}{2\mu^2}|y|^2} dy \end{aligned}$$

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$$= \lim_{\mu\to 0} \int_{\mathbb{R}^n} \left(e^{-\frac{1}{4}L^{dN} g_N |y|^4 - \frac{1}{2}L^{dN}\nu_N |y|^2} + K_N(\Lambda_N, y\mathbf{1})\right) e^{-\frac{L^{dN}}{2\mu^2}|y|^2} dy$$

- $\chi_{g,\nu,N} = L^{-dN}\langle (\sum_x \varphi_x)^2 \rangle_{g,\nu,N}$

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- So likewise,

$$\chi_{g,\nu_c,N} = L^{dN} \lim_{\mu \to 0} \frac{\int_{\mathbb{R}^n} |y|^2 \left(e^{-\frac{1}{4}L^{dN} g_N |y|^4 - \frac{1}{2}L^{dN}\nu_N |y|^2} + K_N(\Lambda_N, y\mathbf{1})\right) e^{-\frac{L^{dN}}{2\mu^2}|y|^2} dy}{\int_{\mathbb{R}^n} \left(e^{-\frac{1}{4}L^{dN} g_N |y|^4 - \frac{1}{2}L^{dN}\nu_N |y|^2} + K_N(\Lambda_N, y\mathbf{1})\right) e^{-\frac{L^{dN}}{2\mu^2}|y|^2} dy}$$

Finite-size susceptibility and the effective potential

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Main issues

1. Convergence of g_j
    - Requires refined control of the dynamical system
2. Integrability of K_j
    - Requires introduction of the Gaussian large field regulator

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Main issues

1. Convergence of g_j
   - Requires refined control of the dynamical system
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   - Requires introduction of the Gaussian large field regulator

$\Rightarrow$ When $d \geq 5$, we have $g_N \to g_\infty > 0$ as $N \to \infty$, thus

$$\chi_{g,\nu_c,N} \sim g_\infty^{-1/2} L^{\frac{d}{2}N} = g_\infty^{-1/2} |\Lambda_N|^{1/2}.$$

Main issue 1: lower bound on g_j

We define the norm such that

$$\|g_j|\varphi|^4\| \asymp L^{-(d-4)j} g_j,$$

so $\|V_j\| \to 0$ does *not* guarantee the convergence of g_j, so we need refined control on V_j. This requires higher order expansion of U_j as a local polynomial, i.e., we need V_j of form

$$V_{j,x}(\varphi) = \frac{1}{4} g_j |\varphi_x|^4 + \frac{1}{2} \nu_j |\varphi_x|^2 + \sum_{m_1} a_{m_1,j} \varphi_x \cdot \nabla^{m_1} \varphi_x + \sum_{m_2} c_{m_2,j} \varphi_x^3 \cdot \nabla^{m_2} \varphi_x,$$

and analyse the dynamical system $(g_j, \nu_j, a_{m_1,j}, c_{m_2,j})_{j \geq 0}$

Main issue 2: integrability of K_j

To obtain the torus scaling limit, we need integrability of K_j against a constant field on Λ_N. In order to obtain this, we require a Gaussian decay condition on K_j: there exits $\kappa > 0$ such that

$$|||K_j(X, \varphi)||| := \left\| e^{\kappa g_j^{1/2} L^{dj/2} \sum_{x \in X} |\varphi_x|^2} K_j(X, \varphi) \right\| = O(\epsilon_j^3),$$

or equivalently

$$|K_j(X, \varphi)| \leq A(X) B(\nabla\varphi, \nabla^2\varphi, \cdots) e^{-\kappa g_j^{1/2} L^{dj/2} \sum_{x \in X} |\varphi_x|^2}$$

for some set function $A(X)$.

Thank you